Advanced Topics in Partial Differential Equations*

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Contents

*These notes are inspired by the lecture series given on the subject by Angeliki Menegak at Imperial College London and supplemented by [2][1].

1 Introduction

1.1 Ordinary Differential Equations

For $F = F(x, y_1, \ldots, y_d)$, the theory of ordinary differential equations seeks to find a differentiable function $u = u(x)$ such that

$$F\left(x, u(x), \ldots, u^{(n)}(x)\right) = 0, \tag{1.1.1}$$

for all $x \in \Omega \subseteq \mathbb{R}$, which is referred to as the domain.

1.2 Partial Differential Equations

Suppose that $u = u(x_0, \ldots, x_d)$ for some $d \geq 1$. Then for fixed $k \geq 1$ and $\Omega \subseteq \mathbb{R}^{d+1}$, an equivalent formulation of (1.1.1) involving partial differential terms $\frac{\partial}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \ldots, \frac{\partial^k u}{\partial x_{i_1} \ldots \partial x_{i_k}}$ is

$$F\left(x_0, \ldots, x_d, \frac{\partial u}{\partial x_0}, \ldots, \frac{\partial u}{\partial x_d}, \ldots, \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}}\right) = F\left(D^k u(x), \ldots, Du(x), u(x), x\right) = 0, \tag{1.2.1}$$

for all $x \in \Omega$, where D denotes the gradient operator and $F : \mathbb{R}^{(d+1)^k} \times \cdots \times \mathbb{R}^{d+1} \times \mathbb{R} \times \Omega \to \mathbb{R}$ is known. Equation (1.2.1) is a k^{th} order partial differential equation and we investigate solving (1.2.1) for $u = u(x_0, \ldots, x_d) : \Omega \to \mathbb{R}$.

Remark 1.2.1. *Partial differential equations are ubiquitous in physical settings. When x_0 can be identified with time, equation* (1.2.1) *is referred to as a parabolic equation. If u is only a function of spatial variables, then* (1.2.1) *is referred to as an elliptic equation.*

To aid notation, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ let

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Moreover, $\alpha! = \alpha_1! \ldots \alpha_d!$ and $x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$. Note the distinction between D^k when $k \in \mathbb{N}$ and when $\alpha \in \mathbb{N}^d$. The former is the k^{th} order gradient operator, whilst the latter is a component of the $|\alpha|^{\text{th}}$ order gradient operator.

Definition 1.2.2.

1. *A linear formulation of* (1.2.1) *has the form*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) = f(x),$$

   where f is given.

2. *A semi-linear formulation of* (1.2.1) *has the form*

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u(x) + a_0\left(D^{k-1} u(x), \ldots, u(x), x\right) = 0.$$

3. *A quasi-linear formulation of* (1.2.1) *has the form*

$$\sum_{|\alpha| = k} a_\alpha\left(D^{k-1} u(x), \ldots, u(x), x\right) D^\alpha u(x) + a_0\left(D^{k-1} u(x), \ldots, u(x), x\right) = 0.$$

4. *A non-linear formulation of* (1.2.1) *depends non-linearly upon the highest-order derivative.*

Remark 1.2.3. *A linear partial differential equation, as given by statement 1 of Definition 1.2.2, has the property that if u and v are solutions to the equation, then so is $au + bv$ for constant a and b.*

Definition 1.2.4. *A system of partial differential equations is a collection of several partial differential equations in several unknown functions. More specifically,*

$$F\left(D^k u(x), \ldots, u(x), x\right) = 0,$$

is a k^{th} order system of partial differential equations, where $u = (u_1, \ldots, u_m) : \Omega \to \mathbb{R}^m$ is the unknown and $F : \mathbb{R}^{m(d+1)^k} \times \cdots \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ is known.

Example 1.2.5.

1. *Poisson's equation is the linear partial differential equation*

$$-\Delta f = g,$$

where $g : \Omega \to \mathbb{R}$ is given and f is unknown. Here $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator. Poisson's equation belongs to the class of elliptic equations.

2. *The heat equation, which introduces a time component to Poisson's equation, is the linear partial differential equation*

$$\partial_t f - \Delta f = g,$$

where $g : I \times \Omega \to \mathbb{R}$ is given and f is unknown. The heat equation belongs to the class of parabolic equations.

3. *The wave equation is the linear partial differential equation*

$$\partial_t^2 f - \Delta f = g,$$

where $g : I \times \Omega \to \mathbb{R}$ is given and f is unknown. The wave equation belongs to the class of hyperbolic equations.

4. *Schrödinger's equation is*

$$i\partial_t f + \Delta f = V f,$$

where $V : I \times \Omega \to \mathbb{R}$ is a potential and $f = f(t, x_1, x_2, x_3)$ is unknown. If V depends on f then we have a non-linear equation, otherwise it is linear. A typical example of the non-linear case is when $V = |f|^2$. Schrödinger's equation is referred to as a dispersive equation.

5. *The incompressible Naiver-Stokes equation is*

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u \\ \nabla_x u = 0. \end{cases}$$

The constant ν describes viscosity, p is the pressure, and $u = u(t, x_1, x_2, x_3)$. The gradient is $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^\top$ such that $\Delta = \nabla \cdot \nabla$.

6. *The Boltzmann equation is*

$$\begin{aligned} \partial_t f + u \cdot \nabla_x f &= Q(f, f) \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left(f\left(v'\right) f\left(v'_*\right) - f(v) f\left(v_*\right)\right) B\left(v - v_*, \sigma\right) \mathrm{d}\sigma \, \mathrm{d}v_*, \end{aligned}$$

where $f = f(t, x_1, x_2, x_3, v_1, v_2, v_3) \geq 0$ is integrable with unit mass, B is the collision kernel, with

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|\sigma}{2}$$

and

$$v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|\sigma}{2}.$$

7. *The reaction-diffusion equation*

$$\underline{u}_t - \Delta u = f(u),$$

where $u = (u_1, u_2)$ and f is unknown.

Solving these equations explicitly requires the specification of boundary conditions.

Definition 1.2.6. *The partial differential equation* (1.2.1) *with space and time boundary conditions is known as a Cauchy problem.*

Definition 1.2.7. *A Cauchy problem is well-posed if a solution exists, is unique and is stable. A solution is stable if it depends continuously on the boundary conditions.*

Remark 1.2.8.

1. *The notion of a solution in Definition 1.2.7 needs to be made precise before a problem can be judged as well-posed. problems may be well-posed under one specification but not another.*

2. *The stability of a problem depends on the topology of the problem. To be stable, perturbations to the initial conditions of a problem need not drastically change the solution it admits.*

Definition 1.2.9. *A solution to* (1.2.1) *is referred to as a classical or strong solution if it exists in $\mathcal{C}^k(\Omega)$ for some $k \in \mathbb{N}$.*

Not all specifications of (1.2.1) admit strong solutions. Indeed, statement 1 of Example 1.2.10 is an explicit specification with no strong solution. However, we can consider so-called weak solutions that loosen the properties a solution to (1.2.1) must satisfy. Statement 2 of Example 1.2.10 shows a way that such a weakening may occur.

Example 1.2.10.

- *Burger's equation is the Cauchy problem*

$$\begin{cases} \partial_t f + f \partial_x f = 0 & x \in \mathbb{R},\, t > 0 \\ f(0, x) = g(x) & x \in \mathbb{R}. \end{cases} \tag{1.2.2}$$

  One can show that if g is at any point decreasing then the solution of (1.2.2) *admits a discontinuity [3].*

- *Let $f \in \mathcal{C}([a, b])$, and consider the Cauchy problem*

$$\begin{cases} -u''(x) + u(x) = f(x) \\ u(a) = u(b) = 0, \end{cases} \tag{1.2.3}$$

  where $u : [a, b] \to \mathbb{R}$. A classical solution to (1.2.3) *would be a solution $u \in \mathcal{C}^2([a, b])$. However, if we* *take $\phi \in \mathcal{C}^1([a,b])$ with $\phi(a) = \phi(b) = 0$, then*

$$\int_a^b -u''(x)\phi(x)\,\mathrm{d}x + \int_a^b u(x)\phi(x)\,\mathrm{d}x = \int_a^b \phi(x)f(x)\,\mathrm{d}x,$$

which we integrate by parts to obtain

$$\int_a^b u'(x)\phi'(x)\,\mathrm{d}x + \int_a^b u(x)\phi(x)\,\mathrm{d}x = \int_a^b f(x)\phi(x)\,\mathrm{d}x. \tag{1.2.4}$$

Equation (1.2.4) *is the weak formulation of* (1.2.3) *and makes sense as long as $u, u' \in L^1(a,b)$ and $f \in L^1(a,b)$. A solution to* (1.2.4) *is referred to as a weak solution.*

2 Ordinary to Partial Differential Equations

In the study of ordinary differential equations, there are a few main theorems that establish the existence and uniqueness of solutions to the differential equations. Stronger regularity conditions of the differential equation lead to stronger results regarding its solutions. The Cauchy-Kovalevskaya theorem, the Picard-Lindelöf theorem, and the Cauchy-Peano theorem progressively reduce the regularity conditions of the differential equation. The Cauchy-Kovalevskaya theorem is the only result which extends to the theory of partial differential equations.

2.1 Analyticity

Definition 2.1.1. *Let $U \subseteq \mathbb{R}$ be open. A function $f \in \mathcal{C}^\infty(U)$ is real-analytic at $x_0 \in U$ if*

$$T(x) = \sum_{n \in \mathbb{N}} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{2.1.1}$$

converges to $f(x)$ for all x in a neighbourhood U_{x_0} of x_0. If f is real-analytic for every $x_0 \in U$, then f is said to be real-analytic on U.

Proposition 2.1.2. *Let $U \subseteq \mathbb{R}$ be open. A function $f \in \mathcal{C}^\infty(U)$ is real-analytic if and only if for all $K \subseteq U$ compact there exists constant $C, r > 0$ such that for all $x \in K$ we have*

$$\left| f^{(n)}(x) \right| \leq C \frac{n!}{r^n}. \tag{2.1.2}$$

Proof. ($\Rightarrow$). Suppose $T(x)$, from (2.1.1), is absolutely-uniformly convergent in $\bar{B}(x_0, r) \subseteq U$. Thus we can let $f(z) = T(z)$ for $z \in \bar{B}(x_0, r) \subseteq \mathbb{C}$. By Cauchy's integral formula it follows that

$$f^{(n)}(x) = \frac{n!}{2\pi i} \int_{|z - x_0| = \frac{r}{2}} \frac{f(z)}{(z - x)^{n+1}} \, \mathrm{d}z$$

for $x \in \bar{B}\left(x_0, \frac{r}{4}\right)$. Thus,

$$\max_{x \in \bar{B}\left(x_0, \frac{r}{4}\right)} \left| f^{(n)}(x) \right| \leq C \frac{n!}{r^n} \|f\|_{L^\infty}.$$

Thus (2.1.2) holds for compact set K of the form $\bar{B}\left(x_0, \frac{r}{4}\right) \subseteq U$. For general compact sets K one considers a finite covering of such closed balls to deduce (2.1.2).
($\Leftarrow$). For $x \in \bar{B}\left(x_0, \frac{r}{2}\right)$, expand the Taylor series of f at x_0 to order n to deduce that

$$f(x) = \sum_{k=0}^{n} f^{(k)}(x_0) \frac{(x - x_0)^k}{k!} + f^{(n+1)}(y_n(x)) \frac{(x - x_0)^{n+1}}{(n+1)!}$$

for some $y_n \in \bar{B}\left(x_0, \frac{r}{2}\right)$. Using (2.1.2) on $K = \bar{B}(x_0, r)$ it follows that

$$\left| f^{(n+1)}(y_n(x)) \frac{(x - x_0)^{n+1}}{(n+1)!} \right| \leq \frac{C}{2^{n+1}} \overset{n \to \infty}{\longrightarrow} 0.$$

Thus the Taylor series is convergent to $f(x)$ for $x \in B\left(x_0, \frac{r}{2}\right)$. $\square$

Example 2.1.3. *Polynomial, exponential and trigonometric functions on $\mathbb{R}$ are analytic.*

Definition 2.1.4. *Let $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^d$ is open. Then f is real-analytic at $x_0 \in U$ if there exists an $r > 0$ and constants $f_\alpha \in \mathbb{R}$ such that*

$$f(x) = \sum_{\alpha \in \mathbb{N}^d} f_\alpha (x - x_0)^\alpha$$

for $|x - x_0| < r$. If f is real-analytic for every $x_0 \in U$, then f is said to be real-analytic on U.

Remark 2.1.5. *In particular, if f is real-analytic near x_0 then it is smooth and the constants $f_\alpha \in \mathbb{R}$ are computed as*

$$f_\alpha = \frac{D^\alpha f(x_0)}{\alpha!}.$$

2.2 General Theorems of Ordinary Differential Equations

Consider the system of ordinary differential equations given by

$$u'(t) = \begin{cases} u_1' = F_1(t, u_1, \dots, u_m) \\ \vdots \\ u_m' = F_m(t, u_1, \dots, u_m) \end{cases} = F(t, u(t)) \tag{2.2.1}$$

for $t \in I \subseteq \mathbb{R}$, where $u = (u_1, \dots, u_m)$. The function $F = (F_1, \dots, F_m)$ is referred to as the vector field and a solution u is referred to as a flow.

Theorem 2.2.1 (Cauchy-Kovalevskaya)**.** *Let $\mathcal{A} \subseteq I \times \mathbb{R}^m$ be open. If F is real-analytic on $\mathcal{A}$ then there exists a unique local analytic $\mathcal{C}^1$ solution. That is, for all $(t_0, u_0) \in \mathcal{A}$ there exists a neighbourhood $U_{t_0} \times U_{u_0} \subseteq \mathcal{A}$ such that* (2.2.1) *has a unique, local, real-analytic solution u on this neighbourhood with $u(t_0) = x_0$.*

To prove Theorem 2.2.1 we consider its form in the case of a scalar ordinary differential equation. Moreover, we assume the existence and uniqueness of a solution by using Theorem 2.2.4 in this setting.

Theorem 2.2.2. *For $b > 0$ suppose that $F : (u_0 - b, u_0 + b) \to \mathbb{R}$ is real-analytic and that $u(t)$ is the unique solution to*

$$u'(t) = F(u(t))$$

with $u(0) = u_0 \in \mathbb{R}$ on some neighbourhood $(-a, a)$, with $u((-a, a)) \subseteq (u_0 - b, u_0 + b)$. Then u is real-analytic on $(-a, a)$.

Proof. Proceed by the method of majorants. Assuming analyticity it follows that

$$\begin{cases} u^{(0)}(t) = F^{(0)}(u(t)) \\ u^{(1)}(t) = F^{(1)}(u(t))u^{(0)}(t) = F^{(1)}(u(t))F^{(0)}(u(t)) \\ u^{(3)}(t) = F^{(2)}(u(t))\left(F^{(0)}(u(t))\right)^2 + \left(F^{(1)}(u(t))\right)^2 F^{(0)}(u(t)) \\ \vdots \end{cases}$$

Through induction one can show that there exists a polynomial p_n such that

$$u^{(n)}(t) = p_n\left(F^{(0)}(u(t)), \dots, F^{(n-1)}(u(t))\right).$$

Therefore,

$$\left|u^{(n)}(0)\right| \le p_n\left(\left|F^{(0)}(0)\right|, \dots, \left|F^{(n-1)}(0)\right|\right).$$

For a majorant function $G \geq 0$ of F with $G^{(n)}(0) \geq \left|F^{(n)}(0)\right|$ it follows that

$$p_n\left(\left|F^{(0)}(0)\right|, \ldots, \left|F^{(n-1)}(0)\right|\right) \leq p_n\left(G^{(0)}(0), \ldots, G^{(n-1)}(0)\right).$$

Letting

$$p_n\left(G^{(0)}(0), \ldots, G^{(n-1)}(0)\right) =: v^{(n)}(0),$$

we obtain an auxiliary differential equation

$$\begin{cases} v'(t) = G(v(t)) \\ v(0) = 0. \end{cases} \tag{2.2.2}$$

Note that $v^{(n)}(0) \geq \left|u^{(n)}(0)\right|$ for all $n \geq 0$ and so if v is real analytic near zero then

$$S_v(t) = \sum_{n \geq 0} \frac{v^{(n)}(0) t^n}{n!}$$

has a positive radius of convergence so that

$$S_u(t) = \sum_{n \geq 0} \frac{\left|u^{(n)}(0)\right| t^n}{n!}$$

has a positive radius of convergence. In other words, for $n \geq 1$ we have

$$0 \leq \left|u^{(n)}(0)\right| \leq C \frac{n!}{\epsilon^n}.$$

As this argument applies uniformly for $t \in [-a', a']$, for some $a' < a$, we can use the uniform growth control on the derivatives of F on $u\left([-a', a']\right)$ to deduce that

$$0 \leq \left|u^{(n)}(t)\right| \leq C \frac{n!}{\epsilon^n}$$

for all $t \in [-a', a']$ and $n \geq 1$. Thus u is analytic in a neighbourhood by Proposition 2.1.2. It remains to find a majorant G of F and show that the corresponding solution v to (2.2.2) is analytic. Let

$$G(z) = C \sum_{n \geq 0} \left(\frac{z}{r}\right)^n = C \frac{1}{1 - \frac{z}{r}} = C\left(\frac{r}{r - z}\right).$$

Then G is real-analytic at $B_r(0)$ and

$$G^{(n)}(0) = C \frac{n!}{r^n}.$$

As F is real-analytic we have

$$\left|F^{(n)}(0)\right| \leq C \frac{n!}{r^n}$$

and so

$$G^{(n)}(0) \geq \left|F^{(n)}(0)\right|$$

for all $n \geq 0$ which means that G is a majorant for F. Through separation of variables the solution to (2.2.2) is

$$v(t) = r - r\sqrt{1 - \frac{2Ct}{r}}$$

which is analytic for $|t| < \frac{r}{2C}$ and so the proof is complete. $\square$

Remark 2.2.3. *The proof of Theorem 2.2.2 can be generalised to the case of Theorem 2.2.1 by recognising that $G(z_1, \ldots, z_n) = (G_1, \ldots, G_m)$ given by*

$$G_k = \frac{Cr}{r - z_1 - \cdots - z_m}$$

for $k = 1, \ldots, m$ is a majorant for F.

Theorem 2.2.4 (Picard-Lindelöf)**.** *Let $\mathcal{L} \subseteq I \times \mathbb{R}^m$ be open. Suppose F is continuous with respect to $t \in I$ and locally Lipschitz in $u \in \mathbb{R}^m$ on $\mathcal{L}$. That is, for every $(t_0, u_0) \in \mathcal{L}$ there exists a neighbourhood $U_{t_0} \times U_{u_0}$ and a constant $c > 0$ such that for all $t \in U_{t_0}$ and for all $u, v \in U_{u_0}$ we have*

$$|F(t, u) - F(t, v)| \leq C|u - v|.$$

Then for all $(t_0, u_0) \in \mathcal{L}$ there exists a neighbourhood $U_{t_0} \times U_{u_0}$ such that (2.2.1) *has a unique local solution in $\mathcal{C}^1$ with $u(t_0) = x_0$.*

Proof. For $u_0 \in \mathcal{L}$ there exists a neighbourhood V of u_0 such that

$$|F(v) - F(w)| < C|v - w|$$

for any $v, w \in V$. In particular, as V is open there exists an $r > 0$ such that $B_r(u_0) \subseteq V$. As f is continuous it is bounded on the compact set $\overline{B_{\frac{r}{2}}(u_0)}$, that is

$$|f(u)| \leq M$$

for all $u \in \overline{B_{\frac{r}{2}}(u_0)}$ and some $M \in \mathbb{R}$. Let

$$C_b := \left\{ w : [-b, b] \to \overline{B_{\frac{r}{2}}(u_0)} \right\} \cap \mathcal{C}^0\left([-b, b]\right),$$

where $b := \min\left(\frac{r}{2M}, \frac{1}{2C}\right)$. Note that C_b is a Banach space with respect to $\|\cdot\|_\infty$. Consider $\Gamma : C_b \to C_b$ given by

$$\Gamma(w)(t) = u_0 + \int_0^t f(w(s))\,\mathrm{d}s.$$

Observe that for $w \in C_b$ and $t \in [-b, b]$ we have

$$\begin{aligned}
\|\Gamma(w) - u_0\|_\infty &= \sup_{t \in [-b,b]} \left| \int_0^t F(w(s))\,\mathrm{d}s \right| \\
&\leq \sup_{t \in [-b,b]} \left(\int_0^t |F(w(s))|\,\mathrm{d}s \right) \\
&\leq \int_0^b M\,\mathrm{d}s \\
&= Mb \\
&\leq M\frac{r}{2M} \\
&= \frac{r}{2},
\end{aligned}$$

and so $\Gamma : C_b \to C_b$ is well-defined. Moreover, for $w_1, w_2 \in C_b$ we have

$$\begin{aligned}
\|\Gamma(w_1) - \Gamma(w_2)\|_\infty &= \sup_{t\in[-b,b]} |\Gamma(w_1)(t) - \Gamma(w_2)(t)| \\
&= \sup_{t\in[-b,b]} \left| \int_0^t F(w_1(s)) - F(w_2(s)) \,\mathrm{d}s \right| \\
&\leq \sup_{t\in[-b,b]} \int_0^t |F(w_1(s)) - F(w_2(s))| \,\mathrm{d}s \\
&\leq \sup_{t\in[-b,b]} \int_0^b C|w_1(s) - w_2(s)| \,\mathrm{d}s \\
&\leq bC\|w_1 - w_2\|_\infty \\
&\leq \frac{1}{2C} C\|w_1 - w_2\|_\infty \\
&= \frac{1}{2}\|w_1 - w_2\|_\infty.
\end{aligned}$$

Therefore, Γ is a contraction. As C_b is a Banach space we can apply Banach's fixed point theorem to deduce that there exists a unique $u \in C_b$ such that $\Gamma(u) = u$. In other words,

$$u = u_0 + \int_0^t F(u(s)) \,\mathrm{d}s,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = F(u(t)).$$

As $u(0) = u_0$ it follows that u solves the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = F(u(t)) \\ u(0) = u_0. \end{cases}$$

As any solution to this Cauchy problem is a fixed of point of Γ it follows that u uniquely solves this problem amongst functions of the form $u : [-b, b] \to \mathbb{R}^n$. ☐

> **Theorem 2.2.5** (Cauchy-Peano)**.** *Let $\mathcal{C} \subseteq I \times \mathbb{R}^m$ be open. If F is continuous with respect to $t \in I$ and $u \in \mathbb{R}^m$ on $\mathcal{C}$, then there exists a local solution in a neighbourhood of (t, u).*

For F let $\mathcal{A} \subseteq I \times \mathbb{R}^m$ be the region where F is real-analytic, $\mathcal{L} \subseteq I \times \mathbb{R}^m$ be the region where F is continuous with respect to $t \in I$ and locally Lipschitz, and $\mathcal{C} \subseteq I \times \mathbb{R}^m$ be the region where F is continuous with respect to $t \in I$ and $u \in \mathbb{R}^m$. Then $\mathcal{A} \subseteq \mathcal{L} \subseteq \mathcal{C} \subseteq I \times \mathbb{R}^m$ such that Theorems 2.2.1, 2.2.4 and 2.2.5 determine the behaviour of solutions as they traverse these regions.

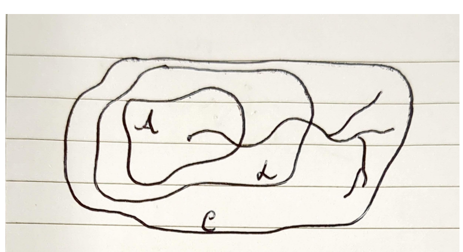


Figure 2.2.1: Regularity on F ensure certain regularity conditions on the solutions of (2.2.1).

Note that if F is linear then $\mathcal{L} = I \times \mathbb{R}^m$ and thus solutions to (2.2.1) exist uniquely on $I \times \mathbb{R}^m$. Therefore, for u to be able to leave $\mathcal{L}$, and to become potentially non-unique, F has to be non-linear.

2.2.1 Local and Global Solutions

Theorems 2.2.1, 2.2.4 and 2.2.5 provide the existence of local solutions. However, it is of interest to understand when local solutions can be extended globally in time. A local solution may not admit a global solution if it blows up in a finite time.

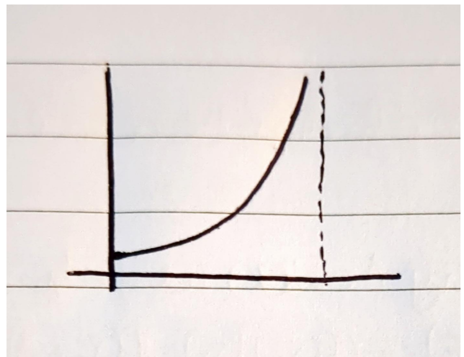


Figure 2.2.2: A solution blowing up in finite time.

Example 2.2.6. *Consider the system*

$$\begin{cases} u'(t) = u^2(t) & t \in \mathbb{R} \\ u(0) = u_0 > 0. \end{cases}$$

Using separation of variables it follows that

$$u = \frac{u_0}{1 - u_0 t}$$

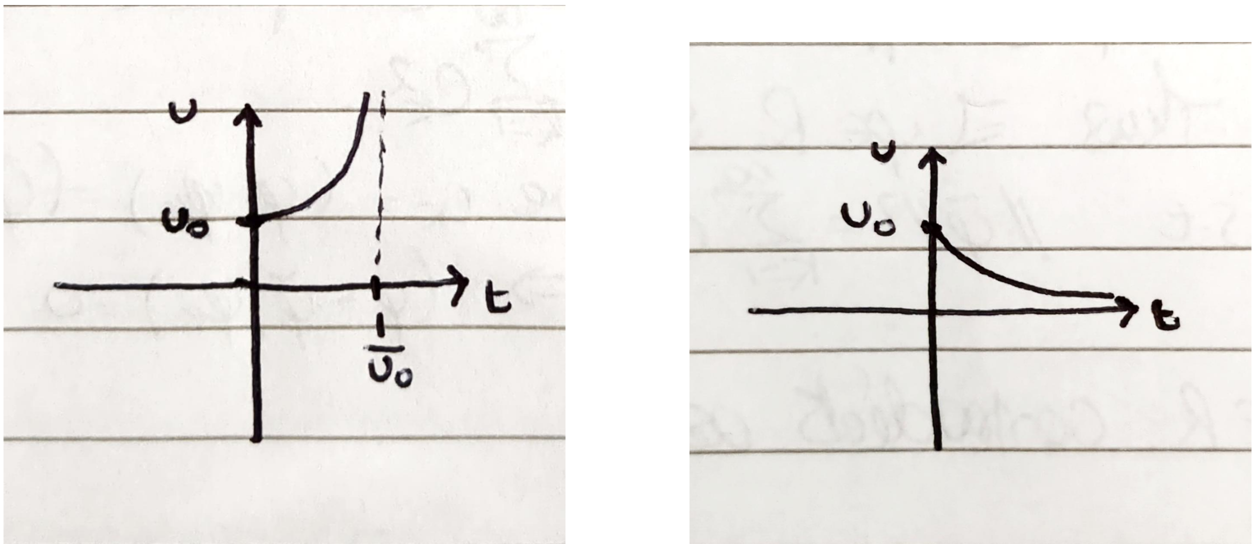
which only exists for $t \in \left[0, \frac{1}{u_0}\right)$*, refer to Figure 2.2.3a. On the other hand,*

$$\begin{cases} u'(t) = -u^2(t) & t \in \mathbb{R} \\ u(0) = u_0 > 0 \end{cases}$$

has the solution

$$u = \frac{u_0}{1 + u_0 t}$$

which exists for all $t \geq 0$*, refer to Figure 2.2.3b.*



(a) The solution to the first system of Example 2.2.6.

(b) The solution to the second system of Example 2.2.6.

Figure 2.2.3: Solutions to the problems of Example 2.2.6.

A criterion on F to avoid blow up in finite time is that F is globally Lipschitz. More specifically, let $I = \mathbb{R}$ and F be a $\mathcal{C}^1$ vector field. If there exists a constant $C > 0$ such that for all $t \in \mathbb{R}$ and $u \in \mathbb{R}^m$ we have

$$|F(t,u)| \leq C\,(1 + |u|)\,,$$

then solutions to

$$u'(t) = F(t, u(t))$$

are global in time.

2.3 Cauchy-Kovalevskaya Theorem for First-Order Partial Differential Equations

2.3.1 Majorants

Definition 2.3.1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be given by*

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha x^\alpha$$

for $f_\alpha \in \mathbb{R}$. Similarly, let $g : \mathbb{R}^d \to \mathbb{R}$ be given by

$$g = \sum_{\alpha \in \mathbb{N}^d} g_\alpha x^\alpha$$

for $g_\alpha \in \mathbb{R}$. Then g majorizes f, written $g \gg f$, if

$$g_\alpha \geq |f_\alpha|$$

for all $\alpha \in \mathbb{N}^d$.

Exercise 2.3.2. *For $x \in \mathbb{R}^d$ and $j \in \mathbb{N}$, show that*

$$(x_1 + \cdots + x_d)^j = \sum_{|\alpha|=j} \binom{|\alpha|}{\alpha} x^\alpha,$$

where $\binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\alpha!}$.

Example 2.3.3. *For $r > 0$ let*

$$g(x) = \frac{r}{r - (x_1 + \cdots + x_d)}$$

for $\|x\| < \frac{r}{\sqrt{d}}$. Then,

$$\begin{aligned} g(x) &= \sum_{k \in \mathbb{N}} \left(\frac{x_1 + \cdots + x_d}{r} \right)^k \\ &= \sum_{k \in \mathbb{N}} \frac{1}{r^k} \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha, \end{aligned}$$

where

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}.$$

Using Cauchy-Schwartz

$$|x_1 + \cdots + x_d| \leq \|x\| \|1\| \leq \sqrt{d} \|x\| < r,$$

and so the series absolutely converges. Thus, the sums can be re-arranged to yield

$$g(x) = \sum_{\alpha \in \mathbb{N}^d} \frac{|\alpha|!}{\alpha! r^{|\alpha|}} x^\alpha.$$

That is,

$$D^\alpha g|_{x=0} = \frac{|\alpha|!}{\alpha! r^{|\alpha|}}$$

for $\alpha \in \mathbb{N}^d$. In conclusion,

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}^d} \frac{|\alpha|!}{\alpha! r^{|\alpha|}} |x|^\alpha &= \sum_{k \geq 0} \left(\frac{|x_1| + \cdots + |x_d|}{r} \right)^k \\ &= \frac{r}{r - (|x_1| + \cdots + |x_n|)} \\ &< \infty, \end{aligned}$$

when $\|x\| < \frac{r}{\sqrt{d}}$.

Lemma 2.3.4. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be given by*

$$f(x) = \sum_{\alpha \in \mathbb{N}^d} f_\alpha x^\alpha$$

for $f_\alpha \in \mathbb{R}$. Similarly, let $g : \mathbb{R}^d \to \mathbb{R}$ be given by

$$g(x) = \sum_{\alpha \in \mathbb{N}^d} g_\alpha x^\alpha$$

for $g_\alpha \in \mathbb{R}$.

1. *Suppose $g \gg f$. If g converges for $\|x\| < r$ then f converges for $\|x\| < r$.*
2. *If f converges for $\|x\| < r$, then there exists a majorant g that converges for $\|x\| \leq \frac{s}{\sqrt{d}}$ where $s \in \left(0, \frac{r}{\sqrt{d}}\right)$.*

Proof.

1. Observe that

$$\begin{aligned}\sum_{|\alpha| \leq k} |f_\alpha x^\alpha| &= \sum_{|\alpha| \leq k} |f_\alpha| |x_1|^{\alpha_1} \dots |x_d|^{\alpha_d} \\ &\leq \sum_{|\alpha| \leq k} g_\alpha |x_1|^{\alpha_1} \dots |x_d|^{\alpha_d} \\ &\leq g(\tilde{x})\end{aligned}$$

where $\tilde{x} = (|x_1|, \dots, |x_d|)$ so that $\|x\| = \|\tilde{x}\|$. Therefore, for $\|x\| < r$ we $\|\tilde{x}\| < r$ and so $g(\tilde{x})$ converges. Thus, the partial sums of f are uniformly bounding and so

$$\sum_{\alpha \in \mathbb{N}^d} |f_\alpha x^\alpha| < \infty,$$

which implies that $|f(x)| < \infty$ and so $f(x)$ converges.

2. Let $s \in \left(0, \frac{r}{\sqrt{d}}\right)$ and consider $y = (s, \dots, s)$ such that $\|y\| = s\sqrt{d} < r$. Then

$$f(y) = \sum_{\alpha \in \mathbb{N}^d} f_\alpha y^\alpha$$

converges. In particular, this means that there exists a $C > 0$ such that $|f_\alpha y^\alpha| \leq C$ and

$$|f_\alpha| \leq \frac{C}{|y^\alpha|} = \frac{C}{|y_1|^{\alpha_1} \dots |y_d|^{\alpha_d}} = \frac{C}{s^{|\alpha|}} \overset{(1)}{\leq} \frac{C}{s^{|\alpha|}} \frac{|\alpha|!}{\alpha!}, \tag{2.3.1}$$

where in (1) the inequality $\alpha! \leq |\alpha|!$ is used. Let

$$g(x) = \frac{Cs}{s - (x_1 + \dots + x_d)} = C \sum_{\alpha \in \mathbb{N}^d} \frac{|\alpha|!}{\alpha! s^{|\alpha|}} x^\alpha.$$

From Example 2.3.3 we know that $g(x)$ converges for $\|x\| < \frac{s}{\sqrt{d}}$. Thus, using (2.3.1), it follows that g is a majorant for f on $\|x\| < \frac{s}{\sqrt{d}}$.

□

Proposition 2.3.5. *Let $x \in \mathbb{R}^d$, and let $f(x) := \sum_{\alpha \in \mathbb{N}^d} f_\alpha x^\alpha$, $g(x) := \sum_{\alpha \in \mathbb{N}^d} g_\alpha x^\alpha$ be formal power series. Then the following statements hold.*

1. *If $g \gg f$ then $D^\beta g \gg D^\beta f$ for every $\beta \in \mathbb{N}^d$.*

2. *If $g \gg f$ and g converges for $|x| \leq r$ then for every $s < r$ we have*

$$\sup_{|x|\leq s} |f(x)| \leq \sup_{|x|\leq s} g(x).$$

Proof.

1. Suppose that

$$D^{\beta} g = \sum_{\alpha \in \mathbb{N}^d} \tilde{g}_{\alpha} x^{\alpha}$$

and

$$D^{\beta} f = \sum_{\alpha \in \mathbb{N}^d} \tilde{f}_{\alpha} x^{\alpha}.$$

If g converges in a neighbourhood of zero, we can differentiate term by term to get

$$\begin{aligned} D^{\beta} g &= \sum_{\alpha \in \mathbb{N}^d} g_{\alpha} D^{\beta} \left(x^{\alpha}\right) \\ &= \sum_{\alpha \in \mathbb{N}^d, \alpha_i \geq \beta_i} g_{\alpha} \frac{\alpha!}{(\alpha - \beta)!} x^{\alpha - \beta}, \end{aligned}$$

where $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_d - \beta_d) \in \mathbb{N}^d$ is well-defined under the assumption that $\alpha_i \geq \beta_i$ for every $i = 1, \ldots, d$. Consequently,

$$\tilde{g}_{\alpha} = \begin{cases} g_{\alpha} \frac{\alpha!}{(\alpha-\beta)!} & \alpha_i \geq \beta_i \text{ for every } i = 1, \ldots, d \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, under the assumption that g converges in a neighbourhood of zero it follows that f also converges in a neighbourhood as $g \gg f$. Thus, we can differentiate term by term to deduce that

$$\tilde{f}_{\alpha} = \begin{cases} f_{\alpha} \frac{\alpha!}{(\alpha-\beta)!} & \alpha_i \geq \beta_i \text{ for every } i = 1, \ldots, d \\ 0 & \text{otherwise.} \end{cases}$$

Hence, as $0 \leq |f_{\alpha}| \leq g_{\alpha}$ for all $\alpha \in \mathbb{N}^d$, it follows that

$$0 \leq \left|\tilde{f}_{\alpha}\right| \leq \tilde{g}_{\alpha}$$

for all $\alpha \in \mathbb{N}^d$. Which means that $D^{\beta} g \gg D^{\beta} f$ for every $\beta \in \mathbb{N}^d$.

2. If $g(x)$ converges for $|x| \leq r$, then $g(x)$ also converges for $|x| \leq s$. Thus, as $g \gg f$ it follows that $f(x)$ converges for $|x| \leq s$. This means that for $|x| \leq s$ the series $f(x) = \sum_{\alpha} f_{\alpha} x^{\alpha}$ and $g(x) = \sum_{\alpha} g_{\alpha} x^{\alpha}$ converge. In particular,

$$\begin{aligned} \sup_{|x|\leq s} |f(x)| &= \sup_{|x|\leq s} \left| \sum_{\alpha \in \mathbb{N}^d} f_{\alpha} x^{\alpha} \right| \\ &\leq \sup_{|x|\leq s} \sum_{\alpha \in \mathbb{N}^d} |f_{\alpha}| |x|^{\alpha} \\ &\overset{g \gg f}{\leq} \sup_{|x|\leq s} \sum_{\alpha \in \mathbb{N}^d} g_{\alpha} |x|^{\alpha} \\ &= \sup_{|x|\leq s} \sum_{\alpha \in \mathbb{N}^d} g_{\alpha} x^{\alpha} \\ &= \sup_{|x|\leq s} g(x). \end{aligned}$$

□

Corollary 2.3.6. *Let $x \in \mathbb{R}^d$, and let $f(x) := \sum_{\alpha \in \mathbb{N}^d} f_\alpha x^\alpha$ be a formal power series. Then f is real analytic at $x = 0$ if and only if $f : B_r(0) \to \mathbb{R}$ is smooth for some $r > 0$ and there exists some constants $C, s, \rho > 0$ such that*

$$\sup_{|x| \leq s} \left| D^\beta f(x) \right| \leq \frac{C|\beta|!}{\rho^{|\beta|}} \tag{2.3.2}$$

for every $\beta \in \mathbb{N}^d$.

Proof. ($\Rightarrow$). As f is real-analytic near zero, for some constants $C > 0$ and $r > 0$ the function

$$g(x) = \sum_{\alpha \in \mathbb{N}^d} \frac{|\alpha|! x^\alpha}{r^{|\alpha|} \alpha!}$$

is a majorant for f and converges to

$$g(x) = \frac{Cr}{r - (x_1 + \cdots + x_d)}$$

for $x \in B_{\frac{r}{\sqrt{d}}}(0)$. Using statement 1 of Proposition 2.3.5 we have $D^\beta g \gg D^\beta f$ and using statement 2 of Proposition 2.3.5, for $0 < s < \frac{r}{\sqrt{d}}$ we have

$$\sup_{|x| \leq s} \left| D^\beta f(x) \right| \leq \sup_{|x| \leq s} \left| D^\beta g(x) \right|. \tag{2.3.3}$$

In particular, we observe that for $|x| \leq s$ we have

$$D^\beta g(x) = D^\beta \left(\frac{Cr}{r - (x_1 + \cdots + x_d)} \right) = \frac{Cr|\beta|!}{(r - (x_1 + \cdots + x_d))^{|\beta|}}.$$

As $r - (x_1 + \cdots + x_d) \geq r - \frac{r}{\sqrt{d}}$ for $|x| \leq s$ we note that

$$\frac{r^{\frac{1}{|\beta|}}}{r - (x_1 + \cdots + x_d)} \leq \frac{1}{\rho}$$

for some $\rho > 0$. In particular, we note that ρ can be chosen independently of β as $r^{\frac{1}{|\beta|}} \leq r$ for all $\beta \in \mathbb{N}^d$. Thus,

$$D^\beta g(x) \leq \frac{C|\beta|!}{\rho^{|\beta|}}$$

for all $|x| \leq s$. Therefore, using (2.3.3) we have

$$\sup_{|x| \leq s} \left| D^\beta f(x) \right| \leq \frac{C|\beta|!}{\rho^{|\beta|}}.$$

($\Leftarrow$). If f is smooth in $B_r(0)$ then it is also smooth in $B_{\tilde{r}}(0)$ for $\tilde{r} < \frac{1}{n} \min(r, s, \rho)$. Therefore, using Taylor's theorem for $x \in B_{\tilde{r}}(0)$ we can write

$$f(x) = \sum_{|\alpha| \leq k-1} \frac{D^\alpha f(0)}{\alpha!} x^\alpha + R_k(x)$$

where

$$R_k(x) = \sum_{|\alpha| = k} \frac{D^\alpha f(\xi)}{\alpha!} x^\alpha$$

for some $\xi \in B_{|x|}(0)$. Observe that

$$\begin{aligned}
|R_k(x)| &= \left| \sum_{|\alpha|=k} \frac{D^\alpha f(\xi)}{\alpha!} x^\alpha \right| \\
&\leq \sum_{|\alpha|=k} \frac{|D^\alpha f(\xi)|}{\alpha!} |x|^\alpha \\
&\overset{(2.3.2)}{\leq} \sum_{|\alpha|=k} \frac{C|\alpha|!}{\alpha! \rho^{|\alpha|}} |x|^\alpha \\
&= \frac{C}{\rho^k} \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} |x|^\alpha \\
&= \frac{C}{\rho^k} \left(|x_1| + \cdots + |x_d|\right)^k \\
&\overset{|x|\leq\tilde{r}}{\leq} C \left(\frac{n\tilde{r}}{\rho}\right)^k \\
&\overset{k\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where convergence follows from the fact that $n\tilde{r} < \rho$. It follows that for $x \in B_{\tilde{r}}(0)$ that

$$f(x) = \sum_{\alpha \in \mathbb{N}^d} \frac{D^\alpha f(0)}{\alpha!} x^\alpha,$$

which means that f is real-analytic in a neighbourhood of zero. $\square$

2.3.2 Cauchy-Kovalevskaya Theorem

Let $x = (x_1, \ldots, x_d) = (x', x_d) \in \mathbb{R}^d$. For $u(x) = \big(u^1(x), \ldots, u^m(x)\big) : \mathbb{R}^d \to \mathbb{R}^m$ let

$$\frac{\partial u}{\partial x_j} := u_{x_j},$$

such that $u_{x_j}^k$ denotes the k^{th} partial derivative of u with respect to x_j. Consider the Cauchy problem

$$\begin{cases} u_{x_d} = \sum_{k=1}^{d-1} \mathbf{B}_j(u, x') u_{x_j} + \mathbf{c}(u, x') & B_r(0) := \left\{x \in \mathbb{R}^d : \|x\| < r\right\} \\ u = 0 & \{x_d = 0\} \cap B_r(0), \end{cases} \tag{2.3.4}$$

where $\mathbf{B}_j = \left(b_j^{kl}\right) : \mathbb{R}^m \times \mathbb{R}^{d-1} \to M_{m\times m}(\mathbb{R})$ and $\mathbf{c} = \left(c^1, \ldots, c^m\right) : \mathbb{R}^m \times \mathbb{R}^{d-1} \to \mathbb{R}^m$. Note how $\mathbf{B}_j\left(u, x'\right)$ and $\mathbf{c}\left(u, x'\right)$ are independent of x_d. In particular, (2.3.4) can be viewed as an evolution equation in the x_d-coordinate in the interior of a ball of radius $r > 0$.

Theorem 2.3.7. *Assume $(\mathbf{B}_j)_{j=1}^{d-1}$ and $\mathbf{c}$ are real-analytic functions. Then there exists an $r > 0$ such that* (2.3.4) *has a real-analytic solution*

$$u = \sum_{\alpha \in \mathbb{N}^d} u_\alpha x^\alpha,$$

which is unique amongst real-analytic functions.

Proof. Step 1: Write each $\mathbf{B}_j$ and $\mathbf{c}$ as power-series.
Without loss of generality suppose that each $\mathbf{B}_j$ and $\mathbf{c}$ is analytic around zero. Hence, we can write

$$\mathbf{B}_j(z, x') = \sum_{\gamma, \delta \in \mathbb{N}^d} \left(\mathbf{B}_j\right)_{\gamma\delta} z^\gamma \left(x'\right)^\delta,$$

and

$$\mathbf{c}(z,x') = \sum_{\gamma,\delta\in\mathbb{N}^d} \mathbf{c}_{\gamma\delta} z^\gamma \left(x'\right)^\delta$$

with $|z|+|x'|<s$ for some $s>0$, where

$$(\mathbf{B}_j)_{\gamma\delta} = \frac{D_z^\gamma D_x^\delta \mathbf{B}_j(0,0)}{\gamma!\delta!}$$

and

$$\mathbf{c}_{\gamma\delta} = \frac{D_z^\gamma D_{x'}^\delta \mathbf{c}(0,0)}{\gamma!\delta!}$$

for $\gamma,\delta\in\mathbb{N}^d$.

Step 2: Find the derivatives of u on $\{x_d=0\}$.

Since $u\equiv 0$ on $\{x_d=0\}$ it follows that

$$u_\alpha = \frac{D^\alpha u(0)}{\alpha!} = 0 \tag{2.3.5}$$

for all $\alpha\in\mathbb{N}^d$ with $\alpha_d=0$. The equation of (2.3.4) written component-wise is

$$u^k_{x_d} = \sum_{j=1}^{d-1}\sum_{l=1}^{m} b_j^{kl}\left(u,x'\right)u^l_{x_j} + c^k\left(u,x'\right)$$

for $k=1,\dots,m$, which when partially differentiated with respect to x_i yields

$$u^k_{x_dx_i} = \sum_{j=1}^{d-1}\sum_{l=1}^{m}\left(b_j^{kl}u^l_{x_ix_j} + \left(b_j^{kl}\right)_{x_i}u^l_{x_l} + \sum_{p=1}^{m}\left(b_j^{kl}\right)_{z_p}u^p_{x_i}u^l_{x_j}\right) + c^k_{x_i} + \sum_{p=1}^{m} c^k_{z_p}u^p_{x_i}.$$

Thus using (2.3.5) we have that $u^k_{x_dx_i}(0) = c^k_{x_i}(0,0)$. By induction it follows that

$$D^\alpha u^k(0) = D^{\alpha'}c^k(0,0)$$

for $\alpha\in\mathbb{N}^d$ of the form $\alpha=(\alpha',1)$. For $\alpha=(\alpha',2)$ note that

$$\begin{aligned}
D^\alpha u^k &= D^{\alpha'}\left(u^k_{x_d}\right)_{x_d}\\
&= D^{\alpha'}\left(\sum_{j=1}^{d-1}\sum_{l=1}^{m} b_j^{kl}u^l_{x_j} + c^k\right)_{x_d}\\
&= D^{\alpha'}\left(\sum_{j=1}^{d-1}\sum_{l=1}^{m}\left(b_j^{kl}u^l_{x_dx_j} + \left(b_j^{kl}\right)_{x_d}u^l_{x_l} + \sum_{p=1}^{m}\left(b_j^{kl}\right)_{z_p}u^p_{x_d}u^l_{x_j}\right) + c^k_{x_d} + \sum_{p=1}^{m}c^k_{z_p}u^p_{x_d}\right).
\end{aligned}$$

Therefore,

$$D^\alpha u^k(0) = D^{\alpha'}\left(\sum_{j=1}^{d-1}\sum_{l=1}^{m} b_j^{kl}u^l_{x_jx_d} + \sum_{p=1}^{m}c^k_{z_p}u^p_{x_d}\right)\Big|_{x=u=0}.$$

More generally, for p^k_α some polynomial with non-negative coefficients

$$D^\alpha u^k(0) = p^k_\alpha\left(\dots, D_z^\gamma D_x^\delta\mathbf{B}_j,\dots,D_z^\gamma D_x^\delta\mathbf{c},\dots,D^\beta\mathbf{u},\dots\right)|_{x=u=0},$$

where $\beta_d\le 1$. In particular,

$$u^k_\alpha = q^k_\alpha\left(\dots,(\mathbf{B}_j)_{\gamma\delta},\dots,\mathbf{c}_{\gamma\delta},\dots,u_\beta,\dots\right)$$

for q_α^k a polynomial with non-negative coefficients and $\beta_d \leq \alpha_d - 1$.

Step 3: Work with a priori majorising functions $\mathbf{B}_j^* \gg \mathbf{B}_j$ and $\mathbf{c}^* \gg \mathbf{c}$.

Suppose that

$$\mathbf{B}_j^* = \sum_{\gamma,\delta\in\mathbb{N}^d} \left(\mathbf{B}_j^*\right)_{\gamma\delta} z^\gamma x^\delta$$

and

$$\mathbf{c}^* = \sum_{\gamma,\delta\in\mathbb{N}^d} \mathbf{c}_{\gamma\delta}^* z^\gamma x^\delta$$

are convergent for $|z| + |x'| < s$. Then,

$$0 \leq \left|(\mathbf{B}_j)_{j\gamma\delta}\right| \leq \left(\mathbf{B}_j^*\right)_{\gamma\delta}$$

and

$$0 \leq |c_{\gamma\delta}| \leq \mathbf{c}_{\gamma\delta}^*.$$

Then consider

$$\begin{cases} u_{x_d}^* = \sum_{j=1}^{d-1} \mathbf{B}_j^* \left(u^*, x'\right) + \mathbf{c}^* \left(u^*, x'\right) & B_r(0) \\ u^* = 0 & \{x_d = 0\} \cap B_r(0). \end{cases} \tag{2.3.6}$$

and suppose it has solution

$$u^* = \sum_{\alpha\in\mathbb{N}^d} u_\alpha^* x^\alpha.$$

Step 4: Show that $u^* \gg u$.

From (2.3.5) we have

$$0 \leq \left|u_\alpha^k\right| \leq \left(u_\alpha^*\right)^k$$

for each $\alpha \in \mathbb{N}^d$ with $\alpha_d = 0$. Assume that

$$0 \leq \left|u_\alpha^k\right| \leq \left(u_\alpha^*\right)^k$$

for each $\alpha \in \mathbb{N}^d$ with $\alpha_d \leq n - 1$. Then for $\alpha \in \mathbb{N}^d$ with $\alpha_d = n$, using the non-negativity of the coefficients of q_α^k, with $\mathbf{B}_j^* \gg \mathbf{B}$ and $\mathbf{c}_j^* \gg \mathbf{c}$ it follows that

$$\begin{aligned} \left|u_\alpha^k\right| &= \left|q_\alpha^k \left(\ldots, (\mathbf{B}_j)_{\gamma\delta}, \ldots, \mathbf{c}_{\gamma\delta}, \ldots, u_\beta, \ldots\right)\right| \\ &\leq q_\alpha^k \left(\ldots, \left|(\mathbf{B}_j)_{\gamma\delta}\right|, \ldots, |\mathbf{c}_{\gamma\delta}|, \ldots, |u_\beta|, \ldots\right) \\ &\overset{(1)}{\leq} q_\alpha^k \left(\ldots, \left(\mathbf{B}_j^*\right)_{\gamma\delta}, \ldots, \mathbf{c}_{\gamma\delta}^*, \ldots, u_\beta^*\right) \\ &= \left(u_\alpha^k\right)^* \end{aligned}$$

where (1) is an application of the inductive hypothesis as $\beta_n \leq \alpha_d - 1 = n - 1$. Therefore, by induction

$$0 \leq \left|u_\alpha^k\right| \leq \left(u_\alpha^*\right)^k$$

for each $\alpha \in \mathbb{N}^d$.

Step 5: Find majorants $\mathbf{B}_j^* \gg \mathbf{B}_j$ and $\mathbf{c}^* \gg \mathbf{c}$.

Using step 1, statement 2 of Lemma 2.3.4 can be applied to yield $\mathbf{B}_j^*$ and $\mathbf{c}_j^*$. In particular,

$$\mathbf{B}_j^* = \frac{Cr}{r - (x_1 + \cdots + x_{d-1}) - (z_1 + \cdots + z_m)} \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ 1 & \ldots & 1 \end{pmatrix}$$

for $j = 1, \ldots, d - 1$ and

$$\mathbf{c}^* = \frac{Cr}{r - (x_1 + \cdots + x_{d-1}) - (z_1 + \cdots + z_m)} (1, \ldots, 1)$$

on $|x'| + |z| < r$ for some $r > 0$. With these, (2.3.6) becomes

$$\begin{cases} u^*_{x_d} = \frac{Cr}{r-(x_1+\cdots+x_{d-1})-(z_1+\cdots+z_m)} \left(\sum_{j=1}^{d-1} \sum_{l=1}^{m} \left(u^*_{x_j} \right)^l + 1 \right) & B_r(0) \\ u^* = 0 & \{x_d = 0\} \cap B_r(0). \end{cases} \tag{2.3.7}$$

Step 6: Show that (2.3.7) has an real-analytic solution u^* and conclude that u is real-analytic.
A solution to (2.3.7) is given by

$$u^* = v^*(1, \ldots, 1)$$

for

$$v^* = \frac{1}{md} \left(r - (x_1 + \cdots + x_{d-1}) - \sqrt{\left(r - (x_1 + \cdots + x_{d-1})^2 \right) - 2mdCrx_d} \right),$$

which is analytic for $|x| < r$ when $r > 0$ sufficiently small. Therefore, using step 4 and statement 1 of Lemma 2.3.4 it follows that u converges for $|x| < r$.
Step 7: Argue that u is a unique real analytic solution to (2.3.4).
As u is real-analytic near zero, by step 6, the Taylor expansions of u_{x_d} and $\sum_{j=1}^{d-1} \mathbf{B}_j(u, x) + \mathbf{c}(u, x)$ agree at zero and on $|x| < r$, thus it is unique. □

Example 2.3.8. *Consider the two-dimensional system, $(u(x, y), v(x, y))$, that satisfies*

$$\begin{cases} u_y = v_x - f \\ v_y = -u_x \\ u = v = 0 & \{y = 0\}, \end{cases} \tag{2.3.8}$$

where $f \in \mathcal{C}^\infty$. Then one can compute all the derivatives of u and v with respect to x and y on $\{y = 0\}$ in the following way. From $u(x, 0) = v(x, 0) = 0$ it follows that

$$(\partial_x)^n\, u(x, 0) = (\partial_x)^n v(x, 0) = 0.$$

Then from $u_y = v_x - f$ we deduce that

$$\partial_y u(x, 0) = -f(x, 0),$$

which implies that,

$$(\partial_x)^n \partial_y u(x, 0) = -(\partial_x)^n f(x, 0).$$

Similarly from $v_y = -u_x$ it follows that

$$\partial_y v(x, 0) = 0$$

and so

$$(\partial_x)^n \partial_y v(x, 0) = 0.$$

Taking the y^{th} derivative of (2.3.8) *we obtain the equations*

$$\begin{cases} u_{yy} = v_{yx} - f_y \\ v_{yy} = -u_{yx}, \end{cases}$$

which on $\{y = 0\}$ become

$$\begin{cases} u_{yy}(x, 0) = -f_y(x, 0) \\ v_{yy}(x, 0) = f_x(x, 0). \end{cases}$$

Hence,

$$(\partial_x)^n \partial_y^2 u(x, 0) = -(\partial_x)^n \partial_y f(x, 0)$$

and

$$(\partial_x)^n \partial_y^2 v(x,0) = (\partial_x)^{n+1} f(x,0).$$

Iterating this process one can obtain all order partial derivatives of u and v with respect to x and y on $\{y=0\}$.

2.4 Cauchy-Kovalevskaya Theorem for Quasi-Linear Partial Differential Equations

Equation (2.3.4) may seem like a rather specific type of partial differential equation, for instance, the right-hand side of (2.3.4) is independent of x_d. However, as the right-hand side can be dependent on u if a dependency on x_d is present we can enlarge the space to $\mathbb{R}^{m+1}$ to accommodate this. More specifically, let $u_{m+1} = x_d$ such that $\frac{\partial u_{m+1}}{\partial x_d} = 1$, which can then be accommodated into (2.3.4) by letting $c^{m+1} = 1$ and for each $j = 1, \ldots, d-1$ letting $b_j^{k,m+1} = 0$ for $k = 1, \ldots, m+1$.

Example 2.4.1. *Consider the partial differential equation*

$$u_{tt} = u u_{xy} - u_{xx} + u_t$$

with $u|_{t=0} = g_0(x,y)$ and $u_t|_{t=0} = g_1(x,y)$ for g_1 and g_2 real-analytic functions.

1. *Let $f(x,y,t) := g_0 + t g_1$, then f is real-analytic with $f|_{t=0} = g_0$ and $f_t|_{t=0} = g_1$. Now set $w = u - f$ such that*
$$w_{tt} = w w_{xy} - w_{xx} + w_t + f w_{xy} + f_{xy} w + F,$$
where $F = f f_{xy} - f_{xx} + f_t$ is independent of w and real-analytic. Observe that $w|_{t=0} = 0$ and $w_t|_{t=0} = 0$. Therefore, we have reduced the system to have trivial boundary conditions.

2. *Consider the transformation $(x,y,t) \mapsto \left(x^1, x^2, x^3\right)$ and let $u = (w, w_x, w_y, w_t) =: \left(u^1, u^2, u^3, u^4\right)$. Then*
$$\begin{cases} \left(u^1\right)_{x_3} = \left(u^1\right)_t = w_t = u^4 \\ \left(u^2\right)_{x_3} = \left(u^2\right)_t = w_{xt} = \left(u^4\right)_{x_1} \\ \left(u^3\right)_{x_3} = \left(u^3\right)_t = w_{yt} = \left(u^4\right)_{x_2} \\ \left(u^4\right)_{x_3} = \left(u^4\right)_t = w_{tt} = u^1 \left(u^2\right)_{x_2} - \left(u^2\right)_{x_1} + u^4 + f\left(u^2\right)_{x_2} + f_{xy} u^1 + F. \end{cases}$$
Therefore,
$$\frac{\partial}{\partial x_3}\begin{pmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{\mathbf{B}_1} \begin{pmatrix} \left(u^1\right)_{x_1} \\ \left(u^2\right)_{x_1} \\ \left(u^3\right)_{x_1} \\ \left(u^4\right)_{x_1} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & f + u^1 & 0 & 0 \end{pmatrix}}_{\mathbf{B}_2} \begin{pmatrix} \left(u^1\right)_{x_2} \\ \left(u^2\right)_{x_2} \\ \left(u^3\right)_{x_2} \\ \left(u^4\right)_{x_2} \end{pmatrix}$$
$$+ \underbrace{\begin{pmatrix} u^4 \\ 0 \\ 0 \\ u^4 + f_{xy} u^1 + F \end{pmatrix}}_{\mathbf{c}}$$
which is of the form (2.3.4) and so Theorem 2.3.7 can be applied.

Remark 2.4.2. *If dealing with a general k^{th} order partial differential equation, then in step 2 of Example 2.4.1 one would take u to be a function in all of the lower-order derivatives. Under some conditions, the arguments of Example 2.4.1 can be generalised to general k^{th} order quasi-linear partial differential equations. In turn, we will arrive at a version of Theorem 2.3.7 that applies to k^{th} order quasi-linear partial differential equations.*

2.4.1 Hypersurfaces

Let Γ be a smooth $(d-1)$-dimensional hypersurface in $U \subseteq \mathbb{R}^d$, with an outward normal vector $n(x) = (n_1(x), \ldots, n_d(x)) : \Gamma \to \mathbb{R}^d$. In particular, denote the j^{th} derivative of u at $x \in \Gamma$ along n by

$$\frac{\partial^j u}{\partial n^j} := \sum_{|\alpha|=j} \binom{j}{\alpha} \frac{\partial^j u}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} n_1^{\alpha_1} \ldots n_d^{\alpha_d} = \sum_{|\alpha|=j} \binom{j}{\alpha} \frac{\partial^\alpha u}{\partial x^\alpha} n^\alpha.$$

Definition 2.4.3. *Let $U \subseteq \mathbb{R}^d$ open. Then ∂U is a $\mathcal{C}^k$-boundary if for all $x_0 \in \partial U$ there exists an $r > 0$ and $\mathcal{C}^k$-function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that*

$$\partial U \cap B_r(x_0) = \{x \in B_r(x_0) : x_d = \gamma(x_1, \ldots, x_{d-1})\}$$

upon relabelling and reorienting the coordinates.

Remark 2.4.4.

1. *Definition 2.4.3 says that locally the boundary of $U \subseteq \mathbb{R}^d$ can be represented by a $\mathcal{C}^k$-function.*
2. *If locally ∂U can be described by a smooth function then ∂U is said to be smooth. Likewise, if locally ∂U can be described by a real-analytic function then ∂U is said to be analytic.*
3. *If Γ, a $(d-1)$-dimensional hypersurface, in $U \subseteq \mathbb{R}^d$ is a $\mathcal{C}^1$-boundary, with outward normal vector n, then for $u \in \mathcal{C}^1\left(\bar{U}\right)$ we have*
$$\frac{\partial u}{\partial n} = n \cdot Du.$$

Example 2.4.5. *Consider the open unit disc*

$$U = \left\{(x, y) : x^2 + y^2 < 1\right\}.$$

Then for $(x_0, y_0) \in \partial U$ the boundary ∂U is represented locally by $(x, \operatorname{sign}(y_0)\gamma(x))$ where $\gamma(x) = \sqrt{1-x^2}$.

Given such a boundary one can consider locally straightening the boundary in one of the dimensions through a change of coordinates. In particular, suppose that we want to flatten the boundary along the x_d axis, then let $\Phi : B(x_0, \epsilon) \to U$ be given by

$$\Phi(x_i) = \begin{cases} x_i & i \leq d-1 \\ x_d - \gamma(x_1, \ldots, x_{d-1}). & \end{cases}$$

With $y := \Phi(x)$ one has
$$\Phi(\partial U \cap B(x_0, \epsilon)) = \{y_d = 0\} \cap U.$$

Moreover, one can construct the reverse transformation

$$\Psi(y_i) = \begin{cases} y_i & i \leq d-1 \\ y_d + \gamma(x_1, \ldots, x_{d-1}), & \end{cases}$$

with $x = \Psi(y)$ so $\Psi = \Phi^{-1}$ and $\det(D\Phi) = \det(D\Psi) = 1$.

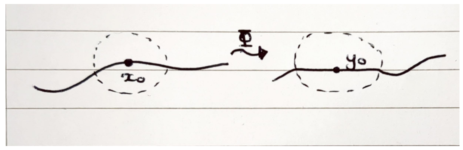


Figure 2.4.1: Flattening a smooth boundary in the neighbourhood of a point using a change of coordinates Φ.

2.4.2 Quasi-Linear Partial Differential Equations

Consider the k^{th} order quasi-linear Cauchy problem given by

$$\begin{cases} \sum_{|\alpha|=k} a_\alpha \left(D^{k-1}u(x), \dots, u(x), x\right) D^\alpha u \\ \qquad\qquad +a_0 \left(D^{k-1}u(x), \dots, u(x), x\right) = 0 & x \in U \subseteq \mathbb{R}^d \\ \frac{\partial^j u}{\partial n^j} = g_j & \Gamma,\ j = 0, \dots, k-1, \end{cases} \tag{2.4.1}$$

where $u : U \to \mathbb{R}$, $g_j : \Gamma \to \mathbb{R}$ are given real-analytic functions for $j = 0, \dots, k-1$ and Γ is assumed to be analytic. Observe that in (2.3.4) we had $\Gamma = \{x_d = 0\}$ and for Example 2.4.1 we had $\Gamma = \{t = 0\}$, each of which are flat boundaries. Moreover, we have seen under a change of coordinates, Γ can be made to be a flat boundary.

Remark 2.4.6. *To compute an analytic solution to* (2.4.1) *it must be the case that all the partial derivatives of u can be determined from* (2.4.1). *In particular, all the partial derivatives of u on Γ should be computable from the boundary conditions. We can use this intuition to arrive at conditions under which an analytic solution to* (2.4.1) *may be determined.*

Definition 2.4.7. *A hypersurface $\Gamma = \{x_d = 0\}$ with the boundary conditions $g_0, \dots, g_{k-1}$ is non-characteristic for* (2.4.1) *at $x_0 \in \Gamma \cap U$ if there exists an open neighbourhood $U_{x_0} \subseteq U$ of x_0 such that*

$$A(x) := a_{(0,\dots,0,k)} \left(D^{k-1}u(x), \dots, u(x), x\right) \neq 0$$

for $x \in \Gamma \cap U_x$

Theorem 2.4.8. *Let $\Gamma = \{x_d = 0\}$ be a non-characteristic for* (2.4.1). *Then if $u \in \mathcal{C}^\infty(U)$ is a solution for* (2.4.1), *the partial derivatives of u on Γ can be determined uniquely by the functions $g_0, \dots, g_{k-1}$ and the coefficients a_α, a_0.*

Proof. Note that a unit normal vector to Γ is given by $n = e_d$, that is the d^{th} standard basis vector $\mathbb{R}^d$ and so $\frac{\partial^j u}{\partial n^j} = \frac{\partial^j u}{\partial x_d}$.

1. As $u = g_0$ on Γ it follows that
$$\frac{\partial u}{\partial x_i} = \frac{\partial g_0}{\partial x_i}$$
for $i \leq d-1$. Using $\frac{\partial u}{\partial x_d} = g_1$ on Γ it follows that $Du = \nabla u$ is determined on Γ.

2. More generally for $\alpha = (\tilde{\alpha}, \alpha_d) \in \mathbb{N}^d$ with $|\alpha_d| \leq k-1$ we have
$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^{\tilde{\alpha}}}{\partial x^{\tilde{\alpha}}} \frac{\partial u}{\partial x_d^{\alpha_d}} = \frac{\partial^{\tilde{\alpha}}}{\partial x^{\tilde{\alpha}}} g_{\alpha_d}.$$
Along with
$$\frac{\partial^j u}{\partial x_d^j} = g_j$$
for $j \leq k-1$ we determine $D^2 u, \dots, D^{k-1}u$ on Γ.

Since, Γ is non-characteristic we can use (2.4.1) to obtain

$$\frac{\partial^k u}{\partial x_d^k} = -\frac{1}{A(x)} \left(\sum_{|\alpha|=k, \alpha_d \leq k-1} a_\alpha \left(D^{k-1}u(x), \ldots, u(x), x\right) + a_0 \left(D^{k-1}u(x), \ldots, u(x), x\right) \right) \tag{2.4.2}$$

on Γ. Thus along with statement 2 we determine $D^k u$ on Γ. Now differentiating (2.4.1) along x_d gives

$$\sum_{|\alpha|=k} a_\alpha \left(D^{k-1}u(x), \ldots, u(x), x\right) D^\alpha u_{x_d}(x) + \tilde{a}_0 \left(D^k u(x), \ldots, u(x), x\right) = 0$$

for $x \in U$, where

$$\begin{aligned} \tilde{a}_0 \left(D^k u(x), \ldots, u(x), x\right) =& \frac{\partial}{\partial x_d} \left(\sum_{|\alpha|=k} a_\alpha \left(D^{k-1}u(x), \ldots, u(x), x\right) \right) D^\alpha u(x) \\ &+ \frac{\partial}{\partial x_d} \left(a_0 \left(D^{k-1}u(x), \ldots, u(x), x\right) \right). \end{aligned}$$

All of this is computable due to previous calculations and is only dependent on the $g_{k-1}, \ldots, g_0$ and coefficients a_α, a_0. Now through similar arguments one obtains

$$\frac{\partial^{k+1} u}{\partial x_l^{k+1}} = -\frac{1}{A(x)} \left(\sum_{|\alpha|=k, \alpha_l \leq k-1} a_\alpha \left(D^{k-1}u(x), \ldots, u(x), x\right) D^\alpha u_{x_d}(x) + \tilde{a}_0 \left(D^k u(x), \ldots, u(x), x\right) \right).$$

Thus, with statement 2 and (2.4.2) one determines $D^{k+1}u$ on Γ. Proceeding inductively it follows that $D^n u$ can be determined on Γ for all $n \in \mathbb{N}$. ☐

Definition 2.4.9. *A hypersurface Γ with boundary conditions $g_0, \ldots, g_{k-1}$ is non-characteristic for* (2.4.1) *if*

$$A(x) := \sum_{|\alpha|=k} a_\alpha \left(D^{k-1}u(x), \ldots, u(x), x\right) n^\alpha \neq 0$$

for all $x \in \Gamma \cap U$ and where n is the normal vector to Γ.

Theorem 2.4.10. *Let Γ be non-characteristic for* (2.4.1)*. Then if $u \in \mathcal{C}^\infty(U)$ is a solution for* (2.4.1)*, the partial derivatives of u on Γ can be determined uniquely by the functions $g_0, \ldots, g_{k-1}$ and the coefficients a_α, a_0.*

Proof. Let $x \in \Gamma$, then there exists smooth maps $\Phi = \left(\Phi^1, \ldots, \Phi^d\right), \Psi = \left(\Psi^1, \ldots, \Psi^d\right) : \mathbb{R}^d \to \mathbb{R}^d$ such that $\Psi = \Phi^{-1}$. Moreover, for some $r > 0$ we have

$$\Phi(\Gamma \cap B(x, r)) \subseteq \{y_d = 0\}$$

where $y = \Phi(x)$. Let $v(y) = u(\Psi(y))$ such that $u(x) = v(\Phi(x))$. Observe that

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^{d} \frac{\partial v}{\partial y_i} \frac{\partial \Phi^j}{\partial x_i}.$$

so that v satisfies the quasi-linear equation

$$\sum_{|\alpha|=k} b_\alpha \left(D^{k-1}v(y), \ldots, v(y), y\right) D^\alpha v + b_0 \left(D^{k-1}v(y), \ldots, v(y), y\right) = 0, \tag{2.4.3}$$

with boundary data

$$\frac{\partial^j v}{\partial y_d^j} = h_j$$

on $\{y_d = 0\}$ for $j = 0, \ldots, k-1$. Note that the coefficients b_α, b_0 and functions h_j are real-analytic as Ψ is real-analytic. Moreover, using the chain rule on $u = v \circ \Phi = v\left(\Phi^1(x), \ldots, \Phi^d(x)\right)$ it follows that when $|\alpha| = k$ and $x \in \Gamma$ we have

$$D^\alpha u = \frac{\partial^k v}{\partial y_d^k}(y)\left(D\Phi(x)\right)^\alpha + \text{lower order terms}$$

where the lower order terms involve only partial derivatives of the order less than $k-1$ in y_d. Therefore, substituting this into (2.4.1) it follows that

$$0 = \sum_{|\alpha|=k} a_\alpha\left(D^{k-1}u(x), \ldots, u(x), x\right)\frac{\partial^k v}{\partial y_d^k}\left(D\Phi^d\right) + \text{lower order terms},$$

thus,

$$b_{(0,\ldots,0,k)} = \sum_{|\alpha|=k} a_\alpha\left(D^{k-1}u(x), \ldots, u(x), x\right)\left(D\Phi^d\right)^\alpha.$$

Since, $\Phi^d(x) = x_d - \gamma\left(x_1, \ldots, x_{d-1}\right)$, it follows that $D\Phi^d$ is parallel to n on Γ. Therefore, $b_{(0,\ldots,0,k)}$ is a non-zero multiple of $\sum_{|\alpha|=k} a_\alpha n^\alpha$ which is non-zero by the assumption that Γ is non-characteristic. Hence, $\{y_d = 0\}$ is a non-characteristic surface of (2.4.3), and so the partial derivatives on of v on $\{y_d = 0\}$ can be determined using Theorem 2.4.8. Then using the reparameterisations and the chain rule we can determine the partial derivatives of u on Γ. □

2.4.3 The Cauchy-Kovalevskaya Theorem

Theorem 2.4.11. *Let Γ be a real-analytic Cauchy surface on $U \subseteq \mathbb{R}^d$. Under real-analytic assumptions on all the coefficients on U, boundary data on Γ, and the non-characteristic condition on Γ there exists a unique, local analytic solution u to* (2.4.1).

Proof.

1. Under the change of coordinates Φ we can transform Γ into the flat boundary $\{x_d = 0\}$. From the proof of Theorem 2.4.10 the hypersurface $\{x_d = 0\}$ is non-characteristic for the transformed equation. Thus without loss of generality, we may assume that $\Gamma = \{x_d = 0\}$.

2. As Γ is non-characteristic, $a_{(0,\ldots,0,k)}(x) \neq 0$ locally on Γ. As $a_{(0,\ldots,0,k)}$ is real-analytic we can divide (2.4.1) by $a_{(0,\ldots,0,k)}$ and maintain the all the real-analytic assumptions. Therefore, without loss of generality, we may assume that $a_{(0,\ldots,0,k)} \equiv 1$.

3. By subtracting appropriate real-analytic functions, we may assume the boundary data is trivial. That is, $g_0, \ldots, g_{k-1} \equiv 0$ on Γ.

4. Let $w = \left(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d}, \ldots, \frac{\partial^{|\beta|}}{\partial x^\beta}\right)$, for all $|\beta| \leq k-1$. In particular, suppose that w has m terms.

With these reductions (2.4.1) is reduced to an equation of the form (2.3.4). Thus one can apply Theorem 2.3.7 to conclude. □

Remark 2.4.12. *Note how the non-characteristic condition is crucial for the reduction of* (2.4.1) *to* (2.3.4). *Physically, this relates to being able to use one of the variables as a time variable to reformulate the problem as an evolution problem. However, in practice finding non-characteristic surfaces is challenging, and poses one of the major obstacles in solving many partial differential equations encountered in physics.*

Example 2.4.13. *For $u = u(t,x) : \mathbb{R}^2 \to \mathbb{R}$, consider the heat equation*

$$\partial_t = \partial_x^2 u \tag{2.4.4}$$

with $u(0,x) = g(x)$. That is, $\Gamma = \{t = 0\}$ with the normal vector $n = e_1 = (1,0)$. Then the non-characteristic condition is $a_{(2,0)} \neq 0$, which does not hold for any Cauchy data g. Therefore, (2.4.4) *is characteristic, and this reflects the fact that* (2.4.4) *cannot be reversed in time. In other words, the Cauchy problem* (2.4.4) *is ill-posed for negative times. More specifically, let*

$$g(x) = \frac{1}{1+x^2} = \sum_{k\in\mathbb{N}} (-1)^k x^{2k}$$

which is clearly real-analytic. Then,

$$\partial_x^{2k} u(0,0) = (-1)^k (2k)!$$

and

$$\partial_x^{2k+1} u(0,0) = 0.$$

Therefore,

$$\partial_t^l \partial_x^{2k} u(0,0) \overset{(2.4.4)}{=} \partial_x^{2k+2l} u(0,0) = (-1)^{k+1}(2k+2l)!.$$

Similarly,

$$\partial_t^l \partial_x^{2k+1} u(0,0) = 0.$$

Thus, if u were real-analytic for negative times one could write

$$\begin{aligned} u(-t,0) &= \sum_{l\in\mathbb{N}} \frac{(-t)^l}{l!} \partial_t^l u(0,0) \\ &= \sum_{l\in\mathbb{N}} t^l \frac{(2l)!}{l!} \\ &\geq \sum_{l\in\mathbb{N}} t^l l^l, \end{aligned}$$

but this does not converge for any $t > 0$. It is in this sense that time cannot be reversed for the heat equation.

Remark 2.4.14.

1. *From Example 2.4.13 we see that characteristic boundary conditions can highlight some key properties of the partial differential equation.*
2. *From Example 2.4.13 it is clear that a necessary condition for an evolution system with boundary conditions on $\Gamma = \{t = 0\}$ to be solvable using Theorem 2.4.11 is that*

$$\partial_t^k u = \sum_{|\alpha|=l} a_\alpha \partial_x^\alpha u$$

with $l \leq k$.

2.4.4 Characteristic Form

Theorem 2.4.11 requires the hypersurface on which the boundary conditions are defined to be non-characteristic. Instead of defining a priori boundary conditions for a partial differential equation, one can instead understand under what conditions boundary conditions are characteristic, and thus also conditions for when boundary conditions are non-characteristic. In turn, one can arrive at a classification of partial differential equations depending on when boundary conditions permit the application of Theorem 2.4.11.

Definition 2.4.15. *Let P be a scalar linear differential operator of order $k \in \mathbb{N}$ given by*

$$Pu := \sum_{|\alpha| \leq k} a_\alpha(x) \partial_x^\alpha u(x)$$

for $u = u(x)$ and $x \in \mathbb{R}^d$. Then the total symbol of P is

$$\sigma(x, \xi) := \sum_{|\alpha| \leq k} a_\alpha(x) \xi^\alpha$$

where $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_d^{\alpha_d}$. The principal symbol, or the characteristic form, of P is

$$\sigma_p(x, \xi) := \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha.$$

Remark 2.4.16. *The non-characteristic condition of a hypersurface Γ, with normal vector n, can be written in terms of the principal symbol as*

$$\sigma_p(x, n(x)) \neq 0$$

for $x \in \Gamma$.

Definition 2.4.17. *The characteristic cone of a partial differential equation at $x \in \mathbb{R}^d$ is*

$$\mathcal{C}_x := \left\{ \xi \in \mathbb{R}^d : \sigma_p(x, \xi) = 0 \right\}.$$

Remark 2.4.18. *Note that a hypersurface Γ, with normal n, is characteristic at a point $x_0 \in \Gamma$ if $n(x_0) \in \mathcal{C}_{x_0}$.*

Definition 2.4.19. *Partial differential equations without real characteristic surfaces are referred to as elliptic equations.*

Example 2.4.20.

1. *Consider Laplace's equation, that is $\Delta u = 0$. Then $\sigma_p(x, \xi) = |\xi|^2$ and so any surface is non-characteristic for Laplace's equation. Indeed the characteristic cone is $\mathcal{C}_x = \{0\}$. Thus, Laplace's equation is elliptic. More generally, let*

   $$L = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

   where $A = (a_{ij})_{i,j=1,\ldots,d}$ is a symmetric matrix and consider

   $$\begin{cases} Lu(x) = 0 & x \in \mathbb{R}^d \\ u = 0 \, , \, \frac{\partial u}{\partial n} = 0 & \Pi_n \end{cases} \tag{2.4.5}$$

   where $\Pi_n := \left\{ x \in \mathbb{R}^d : x \cdot n = 0 \right\}$ for $|n| = 1$. Then the non-characteristic condition for (2.4.5) is

   $$\sum_{i,j=1}^{d} a_{ij} n_i n_j \neq 0.$$

Suppose that the eigenvalues of A are $(\lambda_i)_{i=1,\ldots,d}$, such that by diagonalising A the non-characteristic condition becomes

$$\sum_{i=1}^{d} \lambda_i n_i^2 \neq 0.$$

Consequently, (2.4.5) *is non-characteristic when $\lambda_i > 0$ for each $i \in \{1,\ldots,d\}$ or $\lambda_i < 0$ for each $i \in \{1,\ldots,d\}$.*

2. *Consider Schrödinger's equation,*

$$i\partial_{x_d} + \sum_{i=1}^{d-1} \partial_{x_i}^2 u = 0.$$

Then the characteristic form is

$$\sigma_p(x,\xi) = \xi_1^2 + \cdots + \xi_{d-1}^2.$$

Thus, the only characteristic surfaces are of the form $\{x_d = c\}$ for some constant $c \in \mathbb{R}$.

3. *Consider the wave equation*

$$Lu = -\partial_{x_d}^2 u + \sum_{i=1}^{d-1} \partial_{x_i}^2 u.$$

Then the characteristic form is

$$\sigma_p(x,\xi) = \xi_1^2 + \cdots + \xi_{d-1}^2 - \xi_d^2.$$

Observe that $\sigma_p(x,\xi) = 0$ has non-trivial solutions. Indeed, the characteristic cone in this case is

$$\mathcal{C}_x = \left\{\xi \in \mathbb{R}^d : \xi_d^2 = \xi_1^2 + \cdots + \xi_{d-1}^2\right\},$$

and thus any surface whose normal makes an angle of $\frac{\pi}{4}$ with the $\mathbf{e}_d$ direction is characteristic. The variable x_d represents time.

4. *Consider the transport equation*

$$\sum_{j=1}^{d} c_j(x)\partial_{x_j} u = 0$$

for $u = u(x_1,\ldots,x_d)$. Then the characteristic form is

$$\sigma_p(x,\xi) = \sum_{j=1}^{d} c_j(x)\xi_j$$

with the characteristic cone being

$$c_x = c(x)^{\perp}$$

where $c = (c_1,\ldots,c_d)$ and $x \in \mathbb{R}^d$. Consequently, every characteristic surface is everywhere tangent to $c(x)$. Thus, the transport equation only describes the behaviour of u along a characteristic surface, and what u does in the traversal direction is free. Consequently, the existence of solutions is lost unless the initial condition on the surfaces satisfies certain constraints, and if a solution exists it will not be unique.

Exercise 2.4.21. *Consider*

$$\begin{cases} \left(-\partial_t^2 + \Delta\right) u = 0 \\ u = g_0,\, \partial_t u = g_1 \quad \Gamma \end{cases} \tag{2.4.6}$$

where $u : \mathbb{R}^{1+3} \to \mathbb{R}$ and $\Gamma = \{\phi(x,y,z) = t\}$.

1. *Show that Γ is characteristic everywhere if and only if ϕ obeys the Eikonal equation, namely*

$$|\nabla \phi|^2 = 1.$$

2. *Suppose that $g_0, g_1 \in \mathbb{R}^3 \to \mathbb{R}$ are everywhere real-analytic. Show that in a neighbourhood of $\{t = 0\}$ there exists a unique real analytic solution u to* (2.4.6)*.*

Theorem 2.4.11 has some limitations.

1. Theorem 2.4.11 only provides local solutions. What's more, one has no control over the neighbourhood of the existence of a solution.

2. Theorem 2.4.11 does not guarantee that the corresponding partial differential equation is well-posed. For example, consider
$$u_{xx} + u_{yy} = 0$$
on $\mathbb{R}^2$. With the initial data $u(x, 0) = \cos(kx)$ and $u_y(x, 0) = 0$ on $\{y = 0\}$, through separation of variables on can show that
$$u(x, y) = \cos(kx)\cosh(ky).$$
We note that $u(x, y)$ is a real analytic solution, and therefore it must be the solution provided by Theorem 2.4.11 due to uniqueness. However,
$$\sup_{x\in\mathbb{R}} |u(x, 0)| < 1,$$
and
$$\sup_{x\in\mathbb{R}} |u(x, \epsilon)| = \infty$$
for all $\epsilon > 0$. This shows that there is no continuous dependency between the solution and the initial data, meaning the problem is not well-posed.

3. Theorem 2.4.11 requires strong assumptions on the partial differential equations which limit its applicability.

Exercise 2.4.22. *Consider the equation*

$$u_y - \left(x^2 - c\right) u_{xx} = 0 \tag{2.4.7}$$

for $(x, y) \in \mathbb{R}^2$ and $c > 0$.

1. *Find all the characteristic surfaces to* (2.4.7)*.*

2. *Let* (2.4.7) *be given the initial data*
$$\begin{cases} u(0, y) = \cos(y) \\ u_x(0, y) = 0. \end{cases}$$
For which values of $c > 0$ does (2.4.7) *admit a real-analytic solution in a neighbourhood of $(0, y)$?*

2.5 Solutions to Exercises

Exercise 2.3.2

Solution.

- For $d = 1$ and $j \in \mathbb{N}$ note that $|\alpha| = j$ for $\alpha \in \mathbb{N}^1$ if and only if $\alpha = (j)$. Therefore,
$$(x_1)^j = \frac{j!}{j!}(x_1)^{(j)} = \frac{|(j)|!}{j!}(x_1)^{(j)} = \binom{|(j)|}{(j)}(x_1)^{(j)} = \sum_{\alpha\in\mathbb{N}^1, |\alpha|=j} \binom{|\alpha|}{\alpha} x^\alpha.$$
The case $d = 2$ is the standard binomial theorem.

- Suppose that for $x \in \mathbb{R}^{d-1}$ we have

$$(x_1 + \cdots + x_{d-1})^j = \sum_{\alpha \in \mathbb{N}^{d-1}, |\alpha| = j} \binom{|\alpha|}{\alpha} x^\alpha$$

for every $j \in \mathbb{N}$. Then for $x = (x_1, \ldots, x_{d-1}, x_d) = (x', x_d) \in \mathbb{R}^d$ and $j \in \mathbb{N}$ we have

$$\begin{aligned}
(x_1 + \cdots + x_d)^j &= \sum_{k=0}^{j} \binom{j}{k} (x_1 + \cdots + x_{d-1})^k (x_d)^{j-k} \\
&= \sum_{k=0}^{j} \binom{j}{k} \sum_{\alpha' \in \mathbb{N}^{d-1}, |\alpha'| = k} \binom{|\alpha'|}{\alpha'} (x')^{\alpha'} (x_d)^{j-k} \\
&= \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} \sum_{\alpha' \in \mathbb{N}^{d-1}, |\alpha'| = k} \frac{|\alpha'|!}{\alpha_1! \ldots \alpha_{d-1}!} x^{(\alpha_1, \ldots, \alpha_{d-1}, j-k)} \\
&= \sum_{k=0}^{j} \sum_{\alpha' \in \mathbb{N}^{d-1}, |\alpha'| = k} \frac{j!}{\alpha_1! \ldots \alpha_{d-1}!(j-k)!} x^{(\alpha_1, \ldots, \alpha_{d-1}, j-k)} \\
&= \sum_{\alpha \in \mathbb{N}^d, |\alpha| = j} \frac{|\alpha|}{\alpha!} x^\alpha \\
&= \sum_{|\alpha| = j} \binom{|\alpha|}{\alpha!} x^\alpha.
\end{aligned}$$

Therefore, by induction it follows that for any $d \in \mathbb{N}$, with $x \in \mathbb{R}^d$ we have

$$(x_1 + \cdots + x_d)^j = \sum_{\alpha \in \mathbb{N}^d, |\alpha| = j} \binom{|\alpha|}{\alpha} x^\alpha$$

for every $j \in \mathbb{N}$.

□

Exercise 2.4.21

Solution.

1. The characteristic condition of (2.4.6) is

$$\sum_{i,j=1}^{3} a_{ij} n_i n_j = 0$$

where $A = \operatorname{diag}(-1, 1, 1, 1)$. Writing $\Gamma = \{\phi(x, y, z) - t = 0\} = \{\varphi(t, x, y, z) = 0\}$. The normal to Γ is given by $\nabla \varphi = (1, \nabla \phi) \in \mathbb{R}^{1+3}$. Inserting this into the characteristic condition it follows that

$$-1 + |\nabla \phi|^2 = 0.$$

2. Let $w = (w_1, \ldots, w_5) = (u, u_x, u_y, u_z, u_t)$. Then

$$\begin{cases}
\partial_t w_1 = \partial_t u = w_5 \\
\partial_t w_2 = u_{xt} = \partial_x w_5 \\
\partial_t w_3 = u_{yt} = \partial_y w_5 \\
\partial_t w_4 = u_{zt} = \partial_z w_5 \\
\partial_t w_5 = u_{tt} = u_{xx} + u_{yy} + u_{zz} = \partial_x w_2 + \partial_y w_3 + \partial_z w_4.
\end{cases}$$

Therefore,

$$\partial_t w = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \partial_x w + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \partial_y w + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \partial_z w$$
$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} w$$

with initial conditions

$$w_0 = \begin{pmatrix} g_0 \\ \partial_x g_0 \\ \partial_y g_0 \\ \partial_z g_0 \\ g_1 \end{pmatrix}$$

on $\{t = 0\}$. Let $\tilde{w} = w - w_0$ such that $\partial_t \tilde{w}$ satisfies a matrix equation but with the initial condition that $\tilde{w} = 0$ on $\{t = 0\}$. Now we can apply Theorem 2.3.7 to get the existence and uniqueness of a real-analytic solution in a neighbourhood of zero.

□

Exercise 2.4.22

Solution.

1. As we are working in two dimensions we can parameterise a surface by a curve $\gamma : (a, b) \to \mathbb{R}^2$. More specifically, we let $\gamma(t) = (x(t), y(t))$ so that the normal to the curve is given by $n(t) = (-\dot{y}(t), \dot{x}(t))$. Recall, that γ is a characteristic curve if

   $$\sum_{|\alpha|=2} a_\alpha(x, y) n^\alpha(x, y) = 0$$

   for all $(x, y) \in \gamma$. Consequently, γ is characteristic if

   $$\left(x(t)^2 - c\right)\left(-\dot{y}(t)\right)^2 = 0$$

   for all $t \in (a, b)$. Therefore, the characteristics of (2.4.7) are either

   (a) $\{y = a\}$ for some $a \in \mathbb{R}$,
   (b) $x = c$, or
   (c) $x = -c$.

2. Observe that if $c \neq 0$, then $\{x = 0\}$ is a non-characteristic surface. Moreover, the boundary data is analytic along $\{x = 0\}$ and so we can apply Theorem 2.4.11 to obtain a unique real analytic solution to (2.4.7) in a neighbourhood of $(0, y)$.

□

3 Distributions

3.1 Functions

Example 3.1.1. *Let $U \subseteq \mathbb{R}^d$ be an open bounded set with a smooth boundary. Let $\rho : U \to \mathbb{R}$ be the charge density of U, then the characteristic field satisfies*

$$\begin{cases} \Delta\varphi = \rho & U \\ \varphi = 0 & \partial U. \end{cases} \tag{3.1.1}$$

In such a case, ∂U is referred to as a perfect conductor. Theorem 2.4.11 cannot be applied in this instance as too few boundary data are provided. To be able to solve (3.1.1) *using Theorem 2.4.11 one would also need to know the behaviour of $\nabla\varphi|_{\partial U}$.*

To progress in more general problems, such as Example 3.1.1, we must introduce function space theory to facilitate working with fewer regularity assumptions. Thus, let us recall some prominent function spaces.

- $\mathcal{C}^0\left(\mathbb{R}^d\right)$ is the space of continuous functions on $\mathbb{R}^d$.
- $\mathcal{C}^0_c\left(\mathbb{R}^d\right)$ is the space of continuous functions with compact supports on $\mathbb{R}^d$.
- $\mathcal{C}^r\left(\mathbb{R}^d\right)$ is the space of continuous functions on $\mathbb{R}^d$ with the first r derivatives being continuous. With $\mathcal{C}^r_c\left(\mathbb{R}^d\right)$ as expected.
- $\mathcal{C}^\infty\left(\mathbb{R}^d\right)$ is the space of smooth functions. With $\mathcal{C}^\infty_c\left(\mathbb{R}^d\right)$ as expected.
- $L^p(X)$ for $1 \le p \le \infty$ is the space of functions whose L^p-norm is finite.
- $L^p_{\text{loc}}\left(\mathbb{R}^d\right)$ for $1 \le p \le \infty$ is the space of functions that are in $L^p(K)$ for every compact subset $K \subseteq \mathbb{R}^d$.

3.1.1 Differentiability

Not all function spaces are endowed with a notion of differentiability. This is not ideal when working with differential equations. However, we can still operate under similar regularity conditions by introducing different notions of differentiability. Suppose f is differentiable, then for $\varphi(x) \in \mathcal{C}^\infty_c(\mathbb{R})$, it follows through integration by parts that

$$\int_{\mathbb{R}} f'(x)\varphi(x)\,\mathrm{d}x = -\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x.$$

Where we use the fact that $\varphi(x) = 0$ at $x = \pm\infty$ as φ has compact support.

Exercise 3.1.2. *Show that $-\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x$ is well-defined for $f \in L^1_{\mathit{loc}}(\mathbb{R})$.*

In light of Exercise 3.1.2, for $f \in L^1_{\text{loc}}(\mathbb{R})$ the linear operator $T_f : \mathcal{C}^\infty_c(\mathbb{R}) \to \mathbb{R}$ given by

$$T_f(\varphi) = -\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x.$$

provides an implicit notion of a derivative for f which need not be differentiable in the usual sense.

Definition 3.1.3. *Let $U \subseteq \mathbb{R}^d$ be open, and $f \in L^1_{\mathit{loc}}\left(\mathbb{R}^d\right)$. The α order weak derivative $D^\alpha f \in L_{\mathit{loc}}\left(\mathbb{R}^d\right)$ is such that*

$$\int_U (D^\alpha f)\,\varphi\,\mathrm{d}x = (-1)^{|\alpha|}\int_U f D^\alpha\varphi\,\mathrm{d}x$$

for all $\varphi \in \mathcal{C}^\infty_c(U)$.

Example 3.1.4. *Consider the following Poisson problem,*

$$\begin{cases} -u''(x) + u(x) = f(x) & x \in (a, b) \\ u(a) = u(b) = 0. \end{cases}$$

A strong solution to this problem would be a $u \in \mathcal{C}^2([a, b])$ that satisfies the equation of the problem and the initial conditions. Instead suppose that $\varphi \in \mathcal{C}^1([a, b])$ with $\varphi(a) = \varphi(b) = 0$, and suppose

$$\int_a^b (-u''(x) + u(x))\, \varphi(x)\, \mathrm{d}x = \int_a^b f(x)\varphi(x)\, \mathrm{d}x.$$

Through integration by parts we have

$$\int_a^b -u''(x)\varphi(x)\, \mathrm{d}x = \int_a^b u'(x)\varphi'(x)\, \mathrm{d}x,$$

and so,

$$\int_a^b (u'(x) + u(x))\, \varphi(x)\, \mathrm{d}x = \int_a^b f(x)\varphi(x)\, \mathrm{d}x. \tag{3.1.2}$$

For (3.1.2) *to make sense it is sufficient that $u, u', f \in L^1(a, b)$. A weak solution to our Poisson problem is a function $u \in L^1(a, b)$ that satisfies* (3.1.2).

3.1.2 Support and Convolution

Definition 3.1.5. *For a function f defined on a domain $\Omega \subseteq \mathbb{R}^d$, its support is*

$$\operatorname{supp}(f) := \Omega \cap \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

Remark 3.1.6. *If $\operatorname{supp}(f) \subseteq \Omega$ is compact, then f is said to be compactly supported. The set of smooth and compactly supported functions on Ω is denoted $\mathcal{C}_c^\infty(\Omega)$. The functions of $\mathcal{C}_c^\infty(\Omega)$ are referred to as test functions.*

Example 3.1.7. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be given by*

$$\varphi(x) = \begin{cases} c \exp\left(-\frac{1}{1-|x|^2}\right) & |x| \leq 1 \\ 0 & \textit{otherwise}, \end{cases} \tag{3.1.3}$$

for some $c \in \mathbb{R}$. Note that $\operatorname{supp}(\varphi) = \overline{B_1(0)}$ and φ is smooth, therefore, $\varphi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$.

For functions f, g defined on $\mathbb{R}^d$, their convolution is

$$(f \star g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y)\, \mathrm{d}y = \int_{\mathbb{R}^d} f(y)g(x - y)\, \mathrm{d}y.$$

Theorem 3.1.8 (Young's). *Let $f \in L^p\left(\mathbb{R}^d\right)$ and $g \in L^q\left(\mathbb{R}^d\right)$ for some $p, q \in [1, \infty]$. Then $f \star g \in L^r\left(\mathbb{R}^d\right)$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, and*

$$\|f \star g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

Proposition 3.1.9. *Let $f \in L^1\left(\mathbb{R}^d\right)$ and $g \in L^p\left(\mathbb{R}^d\right)$ for $p \in [1,\infty]$. Then,*

$$\operatorname{supp}(f \star g) \subseteq \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}.$$

3.1.3 Convergence

Convolution can be used to regularise functions. Consider $\varphi \geq 0$ as given by (3.1.3). With $\frac{1}{c} = \int_{B_1(0)} \exp\left(\frac{1}{1-|x|^2}\right) \mathrm{d}x$ we have that $\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x = 1$. In particular, $\epsilon > 0$ let

$$\varphi_\epsilon(x) := \frac{1}{\epsilon^d}\varphi\left(\frac{x}{\epsilon}\right), \tag{3.1.4}$$

such that $\varphi_\epsilon \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, $\operatorname{supp}\left(\varphi_\epsilon\right) = \overline{B_\epsilon(0)}$, and $\int_{\mathbb{R}^d} \varphi_\epsilon(x)\,\mathrm{d}x = 1$. Let $f \in L^p(\Omega)$, for $p \in [1,\infty]$. Setting $f \equiv 0$ in $\mathbb{R}^d \setminus \Omega$ we can think of $f \in L^p\left(\mathbb{R}^d\right)$. Thus we can consider

$$\begin{aligned} f_\epsilon(x) &:= (\varphi_\epsilon \star f)(x) \\ &= \int_\Omega \varphi_\epsilon(x-y) f(y)\,\mathrm{d}y \\ &= \int_{B_\epsilon(0)} \varphi(z) f(x-z)\,\mathrm{d}z. \end{aligned}$$

Lemma 3.1.10.

1. *For $f \in \mathcal{C}^0\left(\mathbb{R}^d\right)$, we have $f_\epsilon \in \mathcal{C}^\infty\left(\mathbb{R}^d\right)$ and $f_\epsilon \to f$ uniformly on every compact set $K \subseteq \mathbb{R}^d$ as $\epsilon \searrow 0$. That is, for $K \subseteq \mathbb{R}^d$ compact we have*

$$\sup_{x \in K} |f_\epsilon(x) - f(x)| \to 0$$

   as $\epsilon \searrow 0$.

2. *For $f \in \mathcal{C}^\infty\left(\mathbb{R}^d\right)$, we have $f_\epsilon \in \mathcal{C}^\infty\left(\mathbb{R}^d\right)$ and for every $\alpha \in \mathbb{N}^d$ we have $D^\alpha f_\epsilon \to D^\alpha f$ uniformly on every compact set $K \subseteq \mathbb{R}^d$ as $\epsilon \searrow 0$.*

3. *For $f \in L^p\left(\mathbb{R}^d\right)$, where $p \in [1,\infty)$, we have $f_\epsilon \in \mathcal{C}^\infty\left(\mathbb{R}^d\right)$. Moreover, $\|f_\epsilon\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}$ with $f_\epsilon \to f$ in L^p-norm as $\epsilon \searrow 0$.*

Proof.

1. Since f is continuous on Ω, it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} f_\epsilon(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_\Omega \varphi_\epsilon(x-y) f(y)\,\mathrm{d}y \\ &= \int_\Omega \frac{\mathrm{d}}{\mathrm{d}x}\varphi_\epsilon(x-y) f(y)\,\mathrm{d}y \\ &= \int_\Omega \varphi_\epsilon'(x) f(y)\,\mathrm{d}y. \end{aligned}$$

   Thus, as $\varphi_\epsilon \in \mathcal{C}^\infty\left(\mathbb{R}^d\right)$, it is clear that $f_\epsilon \in \mathcal{C}^\infty\left(\mathbb{R}^d\right)$. Let $K \subseteq \mathbb{R}^d$ be a compact set. For fixed $\epsilon > 0$ consider $x \in K$. Let $\frac{1}{\epsilon_0} > 0$ be such that $\epsilon_0 > \epsilon$. Then

$$\operatorname{supp}(\varphi_\epsilon) = \overline{B_\epsilon(0)} \subseteq \overline{B_{\epsilon_0}(0)},$$

   and so

$$x - y \in K + \overline{B_{\epsilon_0}(0)} := K',$$

where K' is a compact set. Note that on K', the function f is uniformly continuous, and so there exists a $\theta > 0$ such that for $|y| < \theta$ we have

$$|f(x-y) - f(x)| < \epsilon$$

for all $x \in K$. Observe that for any $\bar{\epsilon} > 0$ we have

$$\begin{aligned}
|(f_{\bar{\epsilon}} - f)(x)| &= \left| \int_{\mathbb{R}^d} \frac{1}{\bar{\epsilon}^d} \varphi\left(\frac{y}{\bar{\epsilon}}\right) (f(x-y) - f(x)) \, \mathrm{d}y \right| \\
&\leq \underbrace{\int_{|y| \geq \theta} \frac{1}{\bar{\epsilon}^d} \varphi\left(\frac{y}{\bar{\epsilon}}\right) |f(x-y) - f(x)| \, \mathrm{d}y}_{I_1} + \underbrace{\int_{|y| \leq \theta} \frac{1}{\bar{\epsilon}^d} \varphi\left(\frac{y}{\bar{\epsilon}}\right) |f(x-y) - f(x)| \, \mathrm{d}y}_{I_2}.
\end{aligned}$$

On the one hand,

$$\begin{aligned}
I_1 &\leq 2\|f\|_{L^\infty(\mathbb{R}^d)} \int_{|y| \geq \theta} \frac{1}{\bar{\epsilon}^d} \varphi\left(\frac{y}{\bar{\epsilon}}\right) \, \mathrm{d}y \\
&\leq 2\|f\|_{L^\infty(\mathbb{R}^d)} \int_{|z| > \frac{\theta}{\bar{\epsilon}}} \varphi(z) \, \mathrm{d}z,
\end{aligned}$$

and so $I_1 = 0$ when $\bar{\epsilon} < \theta$ as $\varphi(z) = 0$ for $|z| \geq 1$. On the other hand, using uniform continuity it follows that

$$\begin{aligned}
I_2 &\leq \epsilon \int_{|y| < \theta} \frac{1}{\bar{\epsilon}^d} \varphi\left(\frac{y}{\bar{\epsilon}}\right) \, \mathrm{d}y \\
&\leq \epsilon(1) \\
&= \epsilon.
\end{aligned}$$

Hence, for $\bar{\epsilon} < \min(\epsilon_0, \theta)$ it follows that

$$|f_\epsilon(x) - f(x)| < \epsilon$$

for all $x \in K$, which means that $f \star \varphi_\epsilon \to f$ uniformly on K as $\epsilon \searrow 0$.

2. As $f \in \mathcal{C}^\infty(\mathbb{R}^d) \subseteq \mathcal{C}^0(\mathbb{R}^d)$, using statement 1 we have that $f_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$. Furthermore, as $D^\alpha(f_\epsilon) \in \mathcal{C}^\infty(\mathbb{R}^d) \subseteq \mathcal{C}^0(\mathbb{R}^d)$ and $D^\alpha f_\epsilon = (D^\alpha f) \star \varphi_\epsilon$, statement 1 can be used to deduce that $D^\alpha f_\epsilon \to D^\alpha f$ uniformly on compacts subsets $K \subseteq \mathbb{R}^d$ as $\epsilon \searrow 0$.

3. Fix $\epsilon > 0$. As $\mathcal{C}_c^0(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d)$ is dense, there exists a $\tilde{f} \in \mathcal{C}_c^0(\mathbb{R}^d)$ such that

$$\|f - \tilde{f}\|_{L^p(\mathbb{R}^d)} < \frac{\epsilon}{3}.$$

Let $\tilde{f}_{\epsilon'} := \tilde{f} \star \varphi_{\epsilon'}$. From statement 1 we know that $\tilde{f}_{\epsilon'} \to \tilde{f}$ uniformly on every compact subset of $\mathbb{R}^d$ as $\epsilon' \searrow 0$. Since,

$$\operatorname{supp}(\tilde{f}_{\epsilon'}) \subseteq \overline{\operatorname{supp}(\tilde{f}) + B_{\epsilon'}(0)},$$

it follows that $\operatorname{supp}(\tilde{f}_{\epsilon'})$ is compact and so

$$\|\tilde{f}_{\epsilon'} - \tilde{f}\|_{L^p(\mathbb{R}^d)} \to 0$$

as $\epsilon' \searrow 0$. Moreover, observe that for arbitrary $g \in L^p(\mathbb{R}^d)$, by letting q be such that $\frac{1}{q} + \frac{1}{p} = 1$, it follows from Hölder's inequality that

$$\begin{aligned}
g_{\epsilon'}(x) &= \int_{\mathbb{R}^d} \varphi_{\epsilon'}(x-y) g(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^d} \varphi_{\epsilon'}(x-y)^{\frac{1}{q}} \varphi_{\epsilon'}(x-y)^{\frac{1}{p}} g(y) \, \mathrm{d}y \\
&\leq \left(\int_{\mathbb{R}^d} \varphi_{\epsilon'}(x-y) \, \mathrm{d}y \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} \varphi_{\epsilon'}(x-y) |g(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}} \\
&\leq \left(\int_{\mathbb{R}^d} \varphi_{\epsilon'}(x-y) |g(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|g_{\epsilon'}\|_{L^p(\mathbb{R}^d)}^p &= \int_{\mathbb{R}^d} |g_{\epsilon'}(x)|^p \, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \varphi_{\epsilon'}(x-y) \, |g(y)|^p \, \mathrm{d}y \right) \mathrm{d}x \\
&\overset{\text{Fubini.}}{=} \int_{\mathbb{R}^d} |g(y)|^p \left(\int_{\mathbb{R}^d} \varphi_{\epsilon'}(x-y) \, \mathrm{d}x \right) \mathrm{d}y \\
&\leq \|g\|_{L^p(\mathbb{R}^d)}^p
\end{aligned} \tag{3.1.5}$$

Using the triangle inequality and (3.1.5) with $f - \tilde{f}$, for sufficiently small ϵ' it follows that

$$\begin{aligned}
\|f_{\epsilon'} - f\|_{L^p(\mathbb{R}^d)} &\leq \left\|\varphi_{\epsilon'} \star (f - \tilde{f})\right\|_{L^p(\mathbb{R}^d)} + \left\|\tilde{f}_{\epsilon'} - \tilde{f}\right\|_{L^p(\mathbb{R}^d)} + \left\|\tilde{f} - f\right\|_{L^p(\mathbb{R}^d)} \\
&\leq \left\|f - \tilde{f}\right\|_{L^p(\mathbb{R}^d)} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

□

Remark 3.1.11.

1. *Note how regardless of the regularity of f, we have $f_\epsilon \in \mathcal{C}^\infty\left(\mathbb{R}^d\right)$. In this sense, φ_ϵ imposes regularity and is thus referred to as a mollified.*
2. *For statement 3 of Lemma 3.1.10 it is necessary not to include $p = \infty$.*

Corollary 3.1.12.

1. *The set of functions $\mathcal{C}^\infty\left(\mathbb{R}^d\right)$ is dense in $\mathcal{C}^0\left(\mathbb{R}^d\right)$ for the topology of uniform convergence on all compact sets.*
2. *The set of function $\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ is dense in $\mathcal{C}^\infty\left(\mathbb{R}^d\right)$ for the topology of uniform convergence of all derivatives on all compact sets.*
3. *The set of functions $\mathcal{C}^\infty\left(\mathbb{R}^d\right)$ is dense in $L^p\left(\mathbb{R}^d\right)$ for $p \in [1, \infty)$.*

Corollary 3.1.13. *Let $v, \tilde{v} \in L^1_{\mathit{loc}}(U)$ be α^{th} order weak derivatives of $u \in L_{\mathit{loc}}(U)$, in the sense of Definition 3.1.3. Then $v = \tilde{v}$ almost everywhere in U.*

Proof. Let $\varphi \in \mathcal{C}_c^\infty(U)$. Then

$$\begin{aligned}
\int_U v(x)\varphi(x) \, \mathrm{d}x &= (-1)^{|\alpha|} \int_U u(x) \left(D^\alpha \varphi\right)(x) \, \mathrm{d}x \\
&= \int_U \tilde{v}(x)\varphi(x) \, \mathrm{d}x
\end{aligned}$$

which implies that

$$\int_U \left(v(x) - \tilde{v}(x)\right) \varphi(x) \, \mathrm{d}x = 0 \tag{3.1.6}$$

for all $\varphi \in \mathcal{C}_c^\infty(U)$. In particular, let $K \subseteq U$ be a compact set and take φ to be the mollifier with support K. With φ_ϵ as given by (3.1.4) it follows that

$$\begin{aligned}(v - \tilde{v})(x) &= \lim_{\epsilon \searrow 0} \left(\left(\varphi_\epsilon \star (v - \tilde{v}) \right)(x) \right) \\ &= \lim_{\epsilon \searrow 0} \int_{y \in \mathbb{R}^d} \varphi_\epsilon(x - y) \, (v - \tilde{v})(y) \, \mathrm{d}y \\ &\overset{(3.1.6)}{=} 0\end{aligned}$$

Therefore, $v - \tilde{v} = 0$ almost everywhere in K. This implies that $v = \tilde{v}$ almost everywhere in U. □

Definition 3.1.14. *Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_c^\infty(\Omega)$ and $f \in \mathcal{C}_c^\infty(\Omega)$. Then $f_n \to f$ as $n \to \infty$ in $\mathcal{C}_c^\infty(\Omega)$ if the following hold.*

1. *There exists a compact set $K \subseteq \Omega$ such that $\operatorname{supp}(f_n) \subseteq K$ for every $n \in \mathbb{N}$.*
2. *For all $\alpha \in \mathbb{N}^d$ we have $D^\alpha f_n \to D^\alpha f$ uniformly on K.*

Remark 3.1.15.

1. *When $f_n \to f$ in $\mathcal{C}_c^\infty(\Omega)$ one often writes $f_n \xrightarrow{\mathrm{D}} f$.*
2. *Note that limits as per Definition 3.1.14 are unique.*

Definition 3.1.16. *The space $\mathcal{C}_c^\infty(\Omega)$ with the convergence provided by Definition 3.1.14 is denoted $\mathrm{D}(\Omega)$.*

3.2 Linear Forms

Definition 3.2.1. *A distribution in $\mathbb{R}^d$ is a linear and continuous form on $\mathrm{D}(\Omega)$. More specifically a distribution is a linear map $T : \mathrm{D}\left(\mathbb{R}^d\right) \to \mathbb{R}$ given by $\varphi \mapsto \langle T, \varphi \rangle$, such that for all sequences $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathrm{D}\left(\mathbb{R}^d\right)$ with $\varphi_n \xrightarrow{\mathrm{D}} \varphi$ we have $\langle T, \varphi_n \rangle \to \langle T, \varphi \rangle$.*

Remark 3.2.2. *The space of distributions is the dual of $\mathrm{D}\left(\mathbb{R}^d\right)$, thus the space of distributions is denoted $\mathrm{D}'\left(\mathbb{R}^d\right)$.*

Example 3.2.3.

1. *Let $f \in L^1_{loc}\left(\mathbb{R}^d\right)$. Let $T_f : \mathrm{D}\left(\mathbb{R}^d\right) \to \mathbb{R}^d$ be given by*

$$T_f(\varphi) = \int_{\mathbb{R}^d} f(x)\varphi(x) \, \mathrm{d}x.$$

Then T_f is well-defined, just as in Exercise 3.1.2.

- *T_f is linear.*

- *Assume* $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathrm{D}\left(\mathbb{R}^d\right)$ *is such that* $\varphi_n \xrightarrow{\mathrm{D}} \varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$. *Then*

$$\begin{aligned}
\left|\langle T_f, \varphi_n\rangle - \langle T_f, \varphi\rangle\right| &= \left|\int_{\mathbb{R}^d} f(x)\left(\varphi_n(x) - \varphi(x)\right)\,\mathrm{d}x\right| \\
&\leq \int_K |f(x)|\,|\varphi_n(x) - \varphi(x)|\,\mathrm{d}x \\
&\leq \|\varphi_n - \varphi\|_{L^\infty(\mathbb{R}^d)} \int_K |f(x)|\,\mathrm{d}x,
\end{aligned}$$

where K *is the compact set provided by statement 1 of Definition 3.1.14. As* $\int_K |f(x)|\,\mathrm{d}x < \infty$ *and* $\|\varphi_n - \varphi\|_{L^\infty(\mathbb{R}^d)} \to 0$ *as* $n \to \infty$ *it follows that*

$$\left|\langle T_f, \varphi_n\rangle - \langle T_f, \varphi\rangle\right| \to 0$$

as $n \to \infty$. *Hence,* T_f *is continuous in* $\mathrm{D}\left(\mathbb{R}^d\right)$ *with respect to the topology induced by Definition 3.1.14. Note the difference to the operator in Exercise 3.1.2 that is not continuous in* $\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ *with respect to the supremum norm.*

Thus $T_f \in \mathrm{D}'\left(\mathbb{R}^d\right)$. *Consequently, we see that* $L^1_{loc}\left(\mathbb{R}^d\right)$ *is continuously embedded in* $\mathrm{D}^1\left(\mathbb{R}^d\right)$*, which we denote* $L^1_{loc}\left(\mathbb{R}^d\right) \hookrightarrow \mathrm{D}'\left(\mathbb{R}^d\right)$. *Similarly, one can show that* $L^p\left(\mathbb{R}^d\right) \hookrightarrow \mathrm{D}'\left(\mathbb{R}^d\right)$ *using Hölder's inequality.*

2. *The Dirac delta at* $x_0 \in \mathbb{R}^d$ *is* $\delta_{x_0} : \mathrm{D}\left(\mathbb{R}^d\right) \to \mathbb{R}$ *where*

$$\delta_{x_0}(\varphi) = \varphi(x_0).$$

The Dirac delta is linear and if $\varphi_n \xrightarrow{\mathrm{D}} \varphi$ *then* $\varphi_n(x_0) \to \varphi(x_0)$*, meaning the Dirac delta is continuous. Therefore,* $\delta_{x_0} \in \mathrm{D}'\left(\mathbb{R}^d\right)$. *However, suppose that* $T_f \equiv \delta_{x_0}$ *for some* $f \in L^1_{loc}\left(\mathbb{R}^d\right)$. *Then for a compact set* $K \subseteq \mathbb{R}^d$ *with* $x_0 \notin K$ *it should be the case that*

$$\int_K f(x)\varphi(x)\,\mathrm{d}x = \int_K \delta_{x_0}(x)\varphi(x)\,\mathrm{d}x = 0$$

for all $\varphi \in \mathrm{D}(K)$. *This implies that* $f(x) = 0$ *almost everywhere on* K. *As* K *was arbitrary, only with the condition that* $x_0 \notin K$*, it follows that* f *is zero almost everywhere on* $\mathbb{R}^d$. *However, this means that*

$$\int_{\mathbb{R}^d} f(x)\varphi(x) = 0 \neq \varphi(x_0)$$

for $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$*, which is a contradiction.*

3. *Let* $f(x) = \frac{1}{x} \notin L^1_{loc}(\mathbb{R})$ *and consider* $\mathrm{pv}\left(\frac{1}{x}\right) : \mathrm{D}(\mathbb{R}) \to \mathbb{R}$ *given by*

$$\mathrm{pv}\left(\frac{1}{x}\right)(\varphi) = \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \frac{\varphi(x)}{x}\,\mathrm{d}x.$$

Using Taylor's formula of φ *around zero, we have*

$$\varphi(x) = \varphi(0) + x\theta(x),$$

for some $\theta \in \mathcal{C}^\infty(\mathbb{R})$. It follows that

$$\begin{aligned}
\left\langle \operatorname{pv}\left(\frac{1}{x}\right), \varphi \right\rangle &= \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \frac{\varphi(0) + x\theta(x)}{x} \,\mathrm{d}x \\
&= \int_{\mathbb{R}\setminus[-1,1]} \frac{\varphi(x)}{x} \,\mathrm{d}x + \lim_{\epsilon \searrow 0} \int_{[-1,1]\setminus[-\epsilon,\epsilon]} \frac{\varphi(0) + x\theta(x)}{x} \,\mathrm{d}x \\
&\overset{(1)}{=} \int_{\mathbb{R}\setminus[-1,1]} \frac{\varphi(x)}{x} \,\mathrm{d}x + \lim_{\epsilon \searrow 0} \int_{[-1,1]\setminus[-\epsilon,\epsilon]} \theta(x) \,\mathrm{d}x.
\end{aligned}$$

where in (1) we use the fact that $\frac{\varphi(0)}{x}$ is odd over the interval $[-1,1] \setminus [-\epsilon,\epsilon]$. Hence, $\operatorname{pv}\left(\frac{1}{x}\right)$ is a distribution.

3.2.1 Convergence and Differentiability

Definition 3.2.4. *Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathrm{D}'\left(\mathbb{R}^d\right)$, then $T_n \to T$ in $\mathrm{D}'\left(\mathbb{R}^d\right)$ if*

$$\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle$$

for all $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$.

Example 3.2.5.

1. *Let $(f_n)_{n\in\mathbb{N}} \subseteq L^1_{loc}\left(\mathbb{R}^d\right)$ and let $f \in L^1_{loc}\left(\mathbb{R}^d\right)$ be such that*

   $$f_n|_K \overset{n\to\infty}{\longrightarrow} f|_K$$

   in $L^1(K)$ for any $K \subseteq \mathbb{R}^d$ compact. It follows that

   $$\begin{aligned}
   |\langle T_{f_n}, \varphi \rangle - \langle T_f, \varphi \rangle| &= \left| \int_{\mathbb{R}^d} \left(f_n(x) - f(x)\right) \varphi(x) \,\mathrm{d}x \right| \\
   &\overset{(1)}{\leq} M \int_{\operatorname{supp}(\varphi)} |f_n(x) - f(x)| \,\mathrm{d}x \\
   &= M \|f_n - f\|_{L^1(\operatorname{supp}(\varphi))}
   \end{aligned}$$

   where (1) follows as $\varphi(x)$ is continuous on a bounded set. As $\operatorname{supp}(\varphi)$ is compact it follows that

   $$\|f_n - f\|_{L^1(\operatorname{supp}(\varphi))} \to 0$$

   as $n \to \infty$ and so $|\langle T_{f_n}, \varphi \rangle - \langle T_f, \varphi \rangle| \to 0$ as $n \to \infty$ for any $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$. Therefore, $T_{f_n} \to T_f$ in $\mathrm{D}'\left(\mathbb{R}^d\right)$.

2. *Let $\varphi(x)$ be as given by* (3.1.3) *with c such that $\int_{\mathbb{R}^d} \varphi(x) \,\mathrm{d}x = 1$. Let φ_ϵ be as given by* (3.1.4)*. Then*

   $$T_{\varphi_\epsilon} \to \delta_0$$

   in $\mathrm{D}'\left(\mathbb{R}^d\right)$ as $\epsilon \searrow 0$.

For $u \in \mathcal{C}^1\left(\mathbb{R}^d\right)$ and $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$ we have

$$\int_{\mathbb{R}^d} \partial_{x_i} u(x) \varphi(x) \,\mathrm{d}x = - \int_{\mathbb{R}^d} u(x) \partial_{x_i} \varphi(x) \,\mathrm{d}x. \tag{3.2.1}$$

Note that $\partial_{x_i}\varphi(x) \in \mathrm{D}\left(\mathbb{R}^d\right)$ and so u on the right-hand side can be replaced by a distribution T. Thus, (3.2.1) can be understood as a characterisation of the derivative of a distribution.

Definition 3.2.6. *Let $T \in \mathrm{D}'\left(\mathbb{R}^d\right)$. Then the generalised derivative of T with respect to x_i is*

$$\left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle = -\left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle$$

for all $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$. More generally, for any $\alpha \in \mathbb{N}^d$ we let

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle$$

for all $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$.

Remark 3.2.7.

1. *If $T = T_u$ for some $u \in \mathcal{C}^1\left(\mathbb{R}^d\right)$, then $\partial_{x_i} T_u = \partial_{x_i} u$. More generally, if $u \in L_{loc}\left(\mathbb{R}^d\right)$, then $D^\alpha T_u$ coincides with Definition 3.1.3.*
2. *Note that a distribution T is infinitely differentiable with commuting derivatives as $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$ is smooth with commuting derivatives.*

Lemma 3.2.8. *If $(T_n)_{n\in\mathbb{N}} \subseteq \mathrm{D}'\left(\mathbb{R}^d\right)$ is such that $T_n \to T$ in $\mathrm{D}'\left(\mathbb{R}^d\right)$. Then $D^\alpha T_n \to D^\alpha T$ in $\mathrm{D}'\left(\mathbb{R}^d\right)$.*

Proof. Let $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$, then for $\alpha \in \mathbb{N}^d$ we have

$$\begin{aligned} \langle D^\alpha T_n, \varphi \rangle &= (-1)^{|\alpha|} \langle T_n, D^\alpha \varphi \rangle \\ &\overset{n\to\infty}{\longrightarrow} (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle \\ &= \langle D^\alpha T, \varphi \rangle, \end{aligned}$$

where the convergence follows as $D^\alpha \varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$ and $T_n \to T$ in $\mathrm{D}'\left(\mathbb{R}^d\right)$. □

Example 3.2.9.

1. *Let*

$$H(x) := \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Then

$$\begin{aligned} \langle H', \varphi \rangle &= -\langle H, \varphi' \rangle \\ &= -\int_{-\infty}^{\infty} H(x)\varphi'(x)\,\mathrm{d}x \\ &= -\int_0^{\infty} \varphi'(x)\,\mathrm{d}x \\ &= \varphi(0), \end{aligned}$$

where in the last step we have used the fact that $\varphi(x) \to 0$ as $x \to \infty$. Hence, $H' = \delta_0$ in the sense of distributions. However, in the pointwise sense, $H' = 0$ almost everywhere, and so we see that the generalised derivative does not coincide with the pointwise derivative. Note also that

$$\langle D^\alpha \delta_{x_0}, \varphi \rangle = (-1)^{|\alpha|} D^\alpha \varphi(x_0)$$

for all $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$.

2. *Let* $f \in \mathcal{C}^k\left(\mathbb{R}^d\right)$ *and* $\alpha \in \mathbb{N}^d$. *Then for all* $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$ *we have*

$$\begin{aligned}\langle D^\alpha T_f, \varphi\rangle &= (-1)^{|\alpha|}\langle T_f, D^\alpha\varphi\rangle \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^d} f(x) D^\alpha\varphi(x)\,\mathrm{d}x \\ &\overset{(1)}{=} \int_{\mathbb{R}^d} D^\alpha f(x)\varphi(x)\,\mathrm{d}x \\ &= \langle T_{D^\alpha f}, \varphi\rangle,\end{aligned}$$

where in (1) *we use reverse integration by parts, which we can do as* $f \in \mathcal{C}^k\left(\mathbb{R}^d\right)$. *Thus it follows that*

$$T_{D^\alpha f} = D^\alpha T_f.$$

Definition 3.2.10. *Let* $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$, $h \in \mathbb{R}^d$ *and* $\lambda \in \mathbb{R}$.

1. *The translation of* φ *by* h *is*
$$\tau_h\varphi(x) = \varphi(x+h)$$
for $x \in \mathbb{R}^d$.
2. *The dilation of* φ *by* λ *is*
$$H_\lambda\varphi(x) = \varphi(\lambda x)$$
for $x \in \mathbb{R}^d$.

Definition 3.2.11. *Let* $T \in \mathrm{D}'\left(\mathbb{R}^d\right)$, $h \in \mathbb{R}^d$ *and* $\lambda \in \mathbb{R}$.

1. *The translation of* T *by* h *is the distribution* $\tau_h T$ *given by*
$$\langle \tau_h T, \varphi\rangle = \langle T, \tau_h\varphi\rangle$$
for $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$.
2. *The dilation of* T *by* λ *is the distribution* $H_\lambda T$ *given by*
$$\langle H_\lambda T, \varphi\rangle = \frac{1}{\lambda^d}\left\langle T, H_{\frac{1}{\lambda}}\varphi\right\rangle$$
for $\varphi \in \mathrm{D}\left(\mathbb{R}^d\right)$.

3.3 Solution to Exercises

Exercise 3.1.2

Solution. Note that $\varphi'(x)$ has compact support, say $K \subseteq \mathbb{R}$. In particular, as $\varphi'(x)$ is continuous on K it is bounded, that is

$$|\varphi'(x)| \leq M$$

for all $x \in K$. Therefore,

$$\begin{aligned}\left|\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x\right| &\leq \int_{\mathbb{R}} |f(x)|\,|\varphi'(x)|\,\mathrm{d}x\\ &\leq M\int_K |f(x)|\,\mathrm{d}x\\ &= M\|f\|_{L^1(K)}\\ &< \infty,\end{aligned}$$

where in the last step we use that $f \in L^1_{\text{loc}}(\mathbb{R})$ and $K \subseteq \mathbb{R}$ is compact. $\square$

4 Sobolev Spaces

It will turn out that Sobolev spaces are the proper setting to apply functional analysis ideas to investigate partial differential equations.

4.1 Hölder Spaces

Definition 4.1.1. *Let $U \subseteq \mathbb{R}^d$ be open. For $k \in \mathbb{N}$ let $\mathcal{C}^k(U)$ denote the set of functions $f : U \to \mathbb{R}$ that are k-times differentiable with $D^\alpha f : U \to \mathbb{R}^{d|\alpha|}$ continuous for every $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$.*

Remark 4.1.2. *As U is open, we can not determine anything about the behaviour of the function of $\mathcal{C}^k(U)$ on ∂U.*

Definition 4.1.3. *Let $U \subseteq \mathbb{R}^d$ be open. For $k \in \mathbb{N}$ let $\mathcal{C}^k\left(\bar{U}\right)$ denote the set of functions $f \in \mathcal{C}^k(U)$ for which $D^\alpha f$ is bounded and uniformly continuous for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$.*

Remark 4.1.4.

1. *Equivalently, $\mathcal{C}^k\left(\bar{U}\right)$ can be seen as the set of functions in $\mathcal{C}^k(U)$ whose derivatives of order less than or equal to k have continuous extensions to the whole of ∂U.*
2. *On $\mathcal{C}^k\left(\bar{U}\right)$ the map $\|\cdot\|_{\mathcal{C}^k(\bar{U})} : \mathcal{C}^k\left(\bar{U}\right) \to \mathbb{R}$ given by*

$$\|f\|_{\mathcal{C}^k(\bar{U})} := \sum_{|\alpha|\leq k} \sup_{x\in U} |D^\alpha f(x)|$$

is well-defined.

Theorem 4.1.5. *The space $\left(\mathcal{C}^k\left(\bar{U}\right), \|\cdot\|_{\mathcal{C}^k(\bar{U})}\right)$ is a Banach space.*

Definition 4.1.6. *A function $f : U \to \mathbb{R}$ is Hölder continuous with index γ if for some $c \in \mathbb{R}$ we have*

$$|f(x) - f(y)| \leq c|x - y|^\gamma$$

for every $x, y \in U$.

Remark 4.1.7. *If f is Hölder continuous with index $\gamma \in (0, 1]$, then f is Lipschitz continuous.*

Definition 4.1.8. *For $U \subseteq \mathbb{R}^d$ open and $\gamma \in (0, 1]$, the 0-Hölder space denoted $\mathcal{C}^{0,\gamma}\left(\bar{U}\right)$ contains functions $u \in \mathcal{C}^0\left(\bar{U}\right)$ such that u is Hölder continuous with index γ.*

Remark 4.1.9. *Note that if u is Hölder continuous with index $\gamma > 1$, then $u' = 0$ which implies that u is constant.*

Exercise 4.1.10. *On $\mathcal{C}^{0,\gamma}\left(\bar{U}\right)$, show that the map $[\cdot]_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)} : \mathcal{C}^{0,\gamma}\left(\bar{U}\right) \to \mathbb{R}$ given by*

$$[u]_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)} := \sup_{x,y\in U} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$

is a semi-norm.

Remark 4.1.11. *The semi-norm of Exercise 4.1.10 can be seen as a computation of the smallest constant $c \in \mathbb{R}$ that satisfies the statement of Definition 4.1.8.*

Proposition 4.1.12. *The map $\|\cdot\|_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)} : \mathcal{C}^{0,\gamma}\left(\bar{U}\right) \to \mathbb{R}$ given by*

$$\|u\|_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)} := [u]_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)} + \|u\|_{\mathcal{C}^{0}\left(\bar{U}\right)}$$

is a norm on $\mathcal{C}^{0,\gamma}\left(\bar{U}\right)$.

Proof. As $\|\cdot\|_{\mathcal{C}^{0}\left(\bar{U}\right)}$ is a norm, using Exercise 4.1.10 it follows that

$$\|\lambda u\|_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)} = |\lambda|\|u\|_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)}$$

for $\lambda \in \mathbb{R}$ and

$$\begin{aligned}\|u+v\|_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)} &\leq [u]_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)} + [v]_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)} + \|u\|_{\mathcal{C}^{0}\left(\bar{U}\right)} + \|v\|_{\mathcal{C}^{0}\left(\bar{U}\right)} \\ &= \|u\|_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)} + \|v\|_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)}.\end{aligned}$$

Suppose $\|u\|_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)} = 0$, then it must be the case that $\|u\|_{\mathcal{C}^{0}\left(\bar{U}\right)} = 0$. As $\|\cdot\|_{\mathcal{C}^{0}\left(\bar{U}\right)}$ is a norm, this happens if and only if $u \equiv 0$. Therefore, $\|\cdot\|_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)}$ is a norm. □

Theorem 4.1.13. *The space $\left(\mathcal{C}^{0,\gamma}\left(\bar{U}\right), \|\cdot\|_{\mathcal{C}^{0,\gamma}\left(\bar{U}\right)}\right)$ is a normed Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}^{0,\gamma}\left(\bar{U}\right)$ be a Cauchy sequence. It follows that $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}^{0}\left(\bar{U}\right)$ is a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{C}^{0}\left(\bar{U}\right)}$. As $\left(\mathcal{C}^{0}\left(\bar{U}\right), \|\cdot\|_{\mathcal{C}^{0}\left(\bar{U}\right)}\right)$ is a Banach space we know that $f_n \to f \in \mathcal{C}^{0}\left(\bar{U}\right)$. For any $(x,y) \in U^2$ with $x \neq y$, let $\delta = |x-y|$. Then as $f_n \to f$ in $\|\cdot\|_{\mathcal{C}^{0}\left(\bar{U}\right)}$ it follows that there exists an $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\delta^\gamma}{2}$$

for any $x \in U$. Therefore, for $n \geq N$ it follows that

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|^\gamma} &\leq \frac{|f(x)-f_n(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)|}{|x-y|^\gamma} \\ &= \frac{|f(x)-f_n(x)| + |f_n(y)-f(y)|}{\delta^\gamma} + \frac{|f_n(x)-f_n(y)|}{|x-y|^\gamma} \\ &\leq \frac{\frac{\delta^\gamma}{2} + \frac{\delta^\gamma}{2}}{\delta^\gamma} + \frac{|f_n(x)-f_n(y)|}{|x-y|^\gamma} \\ &= 1 + \frac{|f_n(x)-f_n(y)|}{|x-y|^\gamma}.\end{aligned}$$

As $(f_n)_{n\in\mathbb{N}} \in \mathcal{C}^{0,\gamma}\left(\bar{U}\right)$ is Cauchy we know that the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded and so $\frac{|f_n(x)-f_n(y)|}{|x-y|} \leq C$ for all $n \in \mathbb{N}$ and $(x,y) \in U^2$. Therefore,

$$\sup_{(x,y)\in U^2, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\gamma} \leq 1 + C$$

and so $f \in \mathcal{C}^{0,\gamma}\left(\bar{U}\right)$. By similar arguments we show that given an $\epsilon > 0$ and $(x,y) \in U^2$ there exits a $N \in \mathbb{N}$ such that for $n \geq N$ we have that

$$\frac{|f(x)-f_n(x)-(f(y)-f_n(y))|}{|x-y|^\gamma} \leq \frac{\epsilon}{2}.$$

Therefore,

$$\sup_{(x,y)\in U^2, x\neq y} \frac{|f(x)-f_n(x)-(f(y)-f_n(y))|}{|x-y|^\gamma} \leq \frac{\epsilon}{2}.$$

Moreover, there exists a $M \in \mathbb{N}$ such that for $n \geq M$ we have that $\|f-f_n\|_{\mathcal{C}^0(\bar{U})} \leq \frac{\epsilon}{2}$ by the fact that $f_n \to f$ in $\|\cdot\|_{\mathcal{C}^0(\bar{U})}$. Therefore,

$$\begin{aligned}
\|f-f_n\|_{\mathcal{C}^{0,\gamma}(\bar{U})} &= \|f-f_n\|_{\mathcal{C}^0(\bar{U})} + \sup_{(x,y)\in U^2, x\neq y} \frac{|f(x)-f_n(x)-(f(y)-f_n(y))|}{|x-y|^\gamma} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

for $n \geq \max(N,M)$. Hence, $f_n \to f$ in $\mathcal{C}^{0,\gamma}\left(\bar{U}\right)$. □

Definition 4.1.14. *Let $\gamma \in (0,1]$. The corresponding k^{th} order Hölder space is*

$$\mathcal{C}^{k,\gamma}\left(\bar{U}\right) := \left\{u \in \mathcal{C}^k\left(\bar{U}\right) : D^\alpha u \in \mathcal{C}^{0,\gamma}\left(\bar{U}\right) \text{ for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha| \leq k\right\}.$$

Remark 4.1.15.

1. *The space $\mathcal{C}^{k,\gamma}\left(\bar{U}\right)$ can be thought of as a space between $\mathcal{C}^k\left(\bar{U}\right)$ and $\mathcal{C}^{k+1}\left(\bar{U}\right)$. In a sense, $u \in \mathcal{C}^{k,\gamma}\left(\bar{U}\right)$ can be seen to be $(k+\gamma)$-times differentiable on $\bar{U}$.*
2. *Note if $U \subseteq \mathbb{R}^d$ is bounded, then for $0 \leq \alpha \leq \beta \leq 1$ and $k = 0, 1, \ldots$ we have*

$$\mathcal{C}^{k,1}\left(\bar{U}\right) \subseteq \mathcal{C}^{k,\beta}\left(\bar{U}\right) \subseteq \mathcal{C}^{k,\alpha}\left(\bar{U}\right) \subseteq \mathcal{C}^{k}\left(\bar{U}\right).$$

Remark 4.1.16. *The map $\|\cdot\|_{\mathcal{C}^{k,\gamma}(\bar{U})} : \mathcal{C}^{k,\gamma}\left(\bar{U}\right) \to \mathbb{R}$ given by*

$$\|u\|_{\mathcal{C}^{k,\gamma}(\bar{U})} := \sum_{\alpha\in\mathbb{N}^d, |\alpha|=k} [D^\alpha u]_{\mathcal{C}^{0,\gamma}(\bar{U})} + \sum_{\alpha\in\mathbb{N}^d, |\alpha|\leq k} \|D^\alpha u\|_{\mathcal{C}^0(\bar{U})}$$

is a norm on $\mathcal{C}^{k,\gamma}\left(\bar{U}\right)$.

Theorem 4.1.17. *The space $\left(\mathcal{C}^{k,\gamma}\left(\bar{U}\right), \|\cdot\|_{\mathcal{C}^{k,\gamma}(\bar{U})}\right)$ is a normed Banach space.*

4.2 Construction

The space $\mathcal{C}^{k,\gamma}\left(\bar{U}\right)$ measures regularity pointwise, whereas Sobolev spaces measure regularity using integrals. Recall, that the space $L^p_{\text{loc}}(U)$ is the space of locally integrable functions. More specifically,

$$L^p_{\text{loc}}(U) = \bigcap_{V \Subset U} L^p(K),$$

where $V \Subset U$ means that there exists a compact set K such that $V \subseteq K \subseteq U$.

Definition 4.2.1. *The space $\mathrm{W}^{k,p}(U)$ consists of functions $u \in L^1_{\text{loc}}(U)$ whose weak derivatives $D^\alpha u$ exist and belong to $L^p(U)$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$.*

Remark 4.2.2.

1. *We require $u \in L^1_{\text{loc}}(U)$ such that u is differentiable in the generalised sense.*
2. *The space $\mathrm{W}^{k,2}(U)$ is usually denoted $\mathrm{H}^k(U)$ to reflect the fact that it is a Hilbert space.*
3. *The map $\|\cdot\|_{\mathrm{W}^{k,p}(U)} : \mathrm{W}^{k,p}(U) \to \mathbb{R}$ given by*

$$\|u\|_{\mathrm{W}^{k,p}(U)} := \begin{cases} \left(\sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq k} \int_U |D^\alpha u|^p\right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq k} \|D^\alpha u\|_{L^\infty(U)} & p = \infty, \end{cases}$$

is referred to as the Sobolev norm on $\mathrm{W}^{k,p}(U)$.

Definition 4.2.3. *For $U \subseteq \mathbb{R}^d$ open, the space $\mathrm{W}^{k,p}_0(U)$ is the completion of $\mathcal{C}^\infty_c(U)$ with respect to $\|\cdot\|_{\mathrm{W}^{k,p}(U)}$. That is,*

$$\mathrm{W}^{k,p}_0(U) := \overline{\mathcal{C}^\infty_c(U)}^{\|\cdot\|_{\mathrm{W}^{k,p}(U)}}.$$

Remark 4.2.4. *As before, H^k_0 is used to denote $\mathrm{W}^{k,2}_0$.*

Example 4.2.5.

1. *Let $U = B_1(0) \subseteq \mathbb{R}^d$. Let $u : U \to \mathbb{R}^d$ be given by*

$$u(x) = \begin{cases} \frac{1}{|x|^\alpha} & x \in U \setminus \{0\} \\ c & x = 0, \end{cases}$$

where $c \in \mathbb{R}$ is some arbitrary value and $\alpha \in \mathbb{R}$ is some fixed constant. The constant c can be arbitrary as Sobolev spaces use an integral measure of regularity, and $\{0\}$ is a set of measure zero and thus insignificant.

- *Note that $u \in L^p(U)$ when*

$$\int_{B_1(0)} \frac{1}{|x|^{\alpha p}} \,\mathrm{d}x = C_d \int_0^1 \frac{1}{r^{\alpha p}} r^{d-1} \,\mathrm{d}r < \infty, \tag{4.2.1}$$

where C_d is some constant that arises as we transition from Cartesian to radial coordinates. Equality (4.2.1) *holds if and only if $\alpha p - d + 1 < 1$. Therefore, $u \in L^p(u)$ if and only if $\alpha p < d$.*

- *When $x \neq 0$ we can compute the derivative of $u(x)$ classically as*

$$D_i u = -\frac{\alpha x_i}{|x|^{\alpha+2}}$$

such that

$$|\nabla u| = \frac{\alpha}{|x|^{\alpha+1}}.$$

Consequently, as weak derivatives are unique, it follows that if u is weakly differentiable then the weak derivative must be given by $D_i u = -\frac{\alpha x_i}{|x|^{\alpha+2}}$. Observe that

$$\begin{aligned} \|\nabla u\|_{L^1(U)} &= \int_{B_1(0)} \frac{\alpha}{|x|^{\alpha+1}} \, \mathrm{d}x \\ &= C_d \int_0^1 \frac{\alpha}{r^{\alpha+1}} r^{d-1} \, \mathrm{d}r \\ &= C_d \int_0^1 \frac{\alpha}{r^{\alpha+2-d}} \, \mathrm{d}r. \end{aligned}$$

Thus, $\|\nabla u\|_{L^1(U)} < \infty$ if and only if $\alpha < d-1$. Now let $\varphi \in \mathcal{C}_c^\infty(U)$, then integrating by parts it follows that

$$-\int_{U \setminus B_\epsilon(0)} u \partial_{x_i} \varphi \, \mathrm{d}x = \int_{U \setminus B_\epsilon(0)} (\partial_{x_i} u) \, \varphi \, \mathrm{d}x - \int_{\partial B_\epsilon(0)} u \varphi n_i \, \mathrm{d}S_i,$$

where $n = (n_1, \dots, n_d)$ is the outward normal vector. Therefore, assuming $\alpha < d-1$ we have that $\|\nabla u\|_{L^1(U)} < \infty$ and so we can apply the dominated convergence theorem to deduce that

$$\int_{U \setminus B_\epsilon(0)} (\partial_{x_i} u) \, \varphi \, \mathrm{d}x \xrightarrow{\epsilon \searrow 0} \int_U (\partial_{x_i} u) \varphi \, \mathrm{d}x,$$

and

$$\int_{U \setminus B_\epsilon(0)} u \partial_{x_i} \varphi \, \mathrm{d}x \xrightarrow{\epsilon \searrow 0} \int_U u \partial_{x_i} \varphi \, \mathrm{d}x.$$

Furthermore, we have

$$\begin{aligned} \left| \int_{\partial B_\epsilon(0)} u \varphi n_i \, \mathrm{d}S_i \right| &\leq \|\varphi\|_{L^\infty} \left(\epsilon^{-\alpha}\right) \underbrace{\left(c\epsilon^{d-1}\right)}_{c \cdot \mathrm{vol}(B_\epsilon(0))} \\ &\leq \tilde{c} \epsilon^{d-1-\alpha} \\ &\xrightarrow{\epsilon \searrow 0} 0, \end{aligned}$$

where convergence follows as we are assuming $\alpha < d-1$. Therefore, for $\alpha < d-1$ the function u is weakly differentiable and thus must have weak derivative $D_i u = -\frac{\alpha x_i}{|x|^{\alpha+2}}$.

- *As before*

$$\|\nabla u\|_{L^p(U)} = C_d \alpha^p \int_0^1 \frac{1}{r^{(\alpha+1)p-d+1}} \, \mathrm{d}r,$$

which if finite if and only if $\alpha < \frac{d-p}{p}$.

Therefore, $u \in \mathrm{W}^{1,p}(U)$ if $\alpha < \frac{d}{p} - 1$ and $u \notin \mathrm{W}^{1,p}(U)$ if $\alpha \geq \frac{d}{p}$. In particular, note that when $p > d$ the condition $\alpha < \frac{d}{p} - 1$ implies that $\alpha < 0$ and so u is continuous on $B_1(0)$.

2. *Let $(r_k)_{k\in\mathbb{N}} \subseteq B_1(0)$ be a dense set. Then let*

$$u(x) := \sum_{k\in\mathbb{N}} \frac{1}{2^k|x-r_k|^k}$$

for $x \in U$. Then u is unbounded in any open subset of U. However, as in statement 1 we have that $u \in W^{1,p}(U)$ when $\alpha < \frac{d}{p} - 1$.

Theorem 4.2.6. *For all $k \in \{0, 1, \dots\}$ and $p \in [1, \infty]$, the Sobolev space $\left(\mathrm{W}^{k,p}(U), \|\cdot\|_{\mathrm{W}^{k,p}(U)}\right)$ is a Banach space.*

Proof.

1. As

$$\|u\|_{\mathrm{W}^{k,p}(U)} = \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$$

it follows that $\|\cdot\|_{\mathrm{W}^{k,p}(U)}$ is positive and homogeneous. Now take $u, v \in \mathrm{W}^{k,p}(U)$, then

$$\begin{aligned}
\|u+v\|_{\mathrm{W}^{k,p}(U)} &= \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha (u+v)|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{|\alpha|\leq k} \|D^\alpha (u+v)\|_{L^p(U)}^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{|\alpha|\leq k} \left(\|D^\alpha u\|_{L^p(U)} + \|D^\alpha v\|_{L^p(U)}\right)^p\right)^{\frac{1}{p}} \\
&\overset{(1)}{\leq} \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p(U)}^p\right)^{\frac{1}{p}} + \left(\sum_{|\alpha|\leq k} \|D^\alpha v\|_{L^p(U)}^p\right)^{\frac{1}{p}} \\
&= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}.
\end{aligned}$$

where (1) is an application of Minkowski's inequality as contextualised on the discrete counting measure. Therefore, $\|\cdot\|_{\mathrm{W}^{k,p}(U)}$ is a norm on $\mathrm{W}^{k,p}(U)$.

2. Observe that

$$\|D^\alpha u\|_{L^p(U)} \leq \|u\|_{\mathrm{W}^{k,p}(U)} \tag{4.2.2}$$

for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$. Let $(u_j)_{j\in\mathbb{N}} \subseteq \mathrm{W}^{k,p}(U)$ be a Cauchy sequence. Then by (4.2.2) it follows that $(D^\alpha u_j)_{j\in\mathbb{N}} \subseteq L^p(U)$ is a Cauchy sequence for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$. Therefore, as $L^p(U)$ is complete, there exists a $u^\alpha \in L^p(U)$ such that $D^\alpha u_j \overset{j\to\infty}{\longrightarrow} u^\alpha$ in $L^p(U)$. Let $u = u^{(0,\dots,0)}$, that is u is the limit of $(u_j)_{j\in\mathbb{N}}$ in L^p. Now let $\varphi \in \mathcal{C}_c^\infty(U)$, then using the definition of a weak derivative we have

$$(-1)^{|\alpha|} \int_U u_j \left(D^\alpha \varphi\right) \, \mathrm{d}x = \int_U \left(D^\alpha u_j\right) \varphi \, \mathrm{d}x$$

for all $j \in \mathbb{N}$. Applying the dominated convergence theorem to both sides it follows that

$$(-1)^{|\alpha|} \int_U u D^\alpha \varphi \, \mathrm{d}x = \int_U u^\alpha \varphi \, \mathrm{d}x.$$

Therefore, by Corollary 3.1.13 it follows that $D^\alpha u = u^\alpha$ in $L^p(U)$, and so $u_j \to u$ in $\mathrm{W}^{k,p}(U)$.

□

4.3 Properties

Having established Sobolev spaces, understanding how functions operate within them will be important. In particular, it will be useful to understand how the properties of the weak derivatives of Sobolev functions.

Theorem 4.3.1. *For $U \subseteq \mathbb{R}^d$, let $u, v \in \mathrm{W}^{k,p}(U)$.*

1. *For all $\alpha, \beta \in \mathbb{N}^d$ with $|\alpha| + |\beta| \le k$ we have $D^\alpha u \in \mathrm{W}^{k-|\alpha|,p}(U)$ and $D^\beta (D^\alpha u) = D^\alpha \left(D^\beta u\right) = D^{\alpha+\beta} u$.*

2. *For $\lambda_1, \lambda_2 \in \mathbb{R}$ we have $\lambda_1 u + \lambda_2 v \in \mathrm{W}^{k,p}(U)$ with*

$$D^\alpha(\lambda_1 u + \lambda_2 v) = \lambda_1 D^\alpha u + \lambda_2 D^\alpha v$$

   for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| \le k$.

3. *If $V \subseteq U$ is open, then $u|_V \in \mathrm{W}^{k,p}(V)$.*

4. *If $\xi \in \mathcal{C}_c^\infty(U)$, then $\xi u \in \mathrm{W}^{k,p}(U)$ and*

$$D^\alpha(\xi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \xi D^{\alpha-\beta} u. \tag{4.3.1}$$

Proof.

1. Note that for $\varphi \in \mathcal{C}_c^\infty(U)$ we have $D^\beta \phi \in \mathcal{C}_c^\infty(U)$. Hence,

$$\begin{aligned}
\int_U (D^\alpha u)\, D^\beta \varphi \,\mathrm{d}x &= (-1)^{|\alpha|} \int_U u D^\alpha D^\beta \varphi \,\mathrm{d}x \\
&= (-1)^{|\alpha|} \int_U u D^{\alpha+\beta} \varphi \,\mathrm{d}x \\
&= (-1)^{|\alpha| + (|\alpha|+|\beta|)} \int_U \left(D^{\alpha+\beta} u\right) \phi \,\mathrm{d}x \\
&= (-1)^{|\beta|} \int_U \left(D^{\alpha+\beta} u\right) \varphi \,\mathrm{d}x.
\end{aligned}$$

   Therefore, by the uniqueness of the weak derivative, we have that $D^\beta (D^\alpha u) = D^{\alpha+\beta} u$. Similarly, $D^\alpha \left(D^\beta u\right) = D^{\alpha+\beta} u$, and so $D^\alpha \left(D^\beta u\right) = D^\beta (D^\alpha u)$. Consequently, for any $\beta \in \mathbb{N}^d$ with $|\beta| \le k - |\alpha|$ we have that $D^\beta (D^\alpha u)$ exists and is in $L^p(U)$ as $|\alpha| + |\beta| \le k$ and $u \in \mathrm{W}^{k,p}(U)$. Therefore, $D^\alpha u \in \mathrm{W}^{k-|\alpha|,p}(U)$.

2. Observe that

$$\begin{aligned}
\int_U (\lambda_1 D^\alpha u + \lambda_2 D^\alpha v)\, \varphi \,\mathrm{d}x &= \lambda_1 \int_U (D^\alpha u)\, \varphi \,\mathrm{d}x + \lambda_2 \int_U (D^\alpha v)\, \varphi \,\mathrm{d}x \\
&= (-1)^{|\alpha|} \lambda_1 \int_U u D^\alpha \varphi \,\mathrm{d}x + (-1)^{|\alpha|} \lambda_2 \int_U v D^\alpha \varphi \,\mathrm{d}x \\
&= (-1)^{|\alpha|} \int_U (\lambda_1 u + \lambda_2 v)\, D^\alpha \varphi \,\mathrm{d}x.
\end{aligned}$$

   Therefore, by the uniqueness of the weak derivative, it follows that

$$D^\alpha (\lambda_1 u + \lambda_2 v) = \lambda_1 D^\alpha u + \lambda_2 D^\alpha v.$$

   In particular, this means that for $|\alpha| \le k$, the weak derivative $D^\alpha (\lambda_1 u + \lambda_2 v)$ exists. Moreover,

$$\|D^\alpha (\lambda_1 u + \lambda_2 v)\|_{L^p(U)} \le |\lambda_1| \, \|D^\alpha u\|_{L^p(U)} + |\lambda_2| \, \|D^\alpha v\|_{L^p(U)} < \infty,$$

   which means that $\lambda_1 u + \lambda_2 v \in \mathrm{W}^{k,p}(U)$.

3. Any $\varphi \in \mathcal{C}_c^\infty(V)$ can thought of as $\varphi \in \mathcal{C}_c^\infty(U)$ by letting $\varphi(x) = 0$ for $x \in U \setminus V$. For $\alpha \in \mathbb{N}^d$ with $|\alpha| \le k$, let $D^\alpha u$ be the weak derivative of u on U. Then

$$\begin{aligned}\int_V (D^\alpha u)\,\varphi\,\mathrm{d}x &= \int_U (D^\alpha u)\,\varphi\,\mathrm{d}x \\ &= (-1)^{|\alpha|}\int_U u D^\alpha \varphi\,\mathrm{d}x \\ &= (-1)^{|\alpha|}\int_V u D^\alpha \varphi\,\mathrm{d}x.\end{aligned}$$

Therefore, the α-order weak derivative of $u|_V$ is $D^\alpha u|_V$. Hence, as

$$\|D^\alpha u\|_{L^p(V)} \le \|D^\alpha u\|_{L^p(U)}$$

it follows that $u|_V \in \mathrm{W}^{k,p}(V)$.

4. Proceed by induction of $|\alpha|$.

   - For $|\alpha| = 1$, let $\varphi \in \mathcal{C}_c^\infty(U)$. Then using the product rule of ξ, φ it follows that

   $$\begin{aligned}\int_U \xi u D^\alpha \varphi\,\mathrm{d}x &= \int_U u D^\alpha(\xi\varphi) - u\,(D^\alpha \xi)\,\varphi\,\mathrm{d}x \\ &= -\int_U (\xi D^\alpha u + u D^\alpha \xi)\,\varphi\,\mathrm{d}x.\end{aligned}$$

   Thus, $D^\alpha(\xi u) = \xi D^\alpha u + u D^\alpha \xi$.

   - Assume that (4.3.1) holds for all $\xi \in \mathcal{C}_c^\infty(U)$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| \le l < k$. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| = l+1$, such that $\alpha = \beta + \gamma$ for $\beta, \gamma \in \mathbb{N}^d$ with $|\beta| = l$ and $|\gamma| = 1$. Then for $\varphi \in \mathcal{C}_c^\infty(U)$ it follows that

   $$\begin{aligned}\int_U \xi u D^\alpha \varphi\,\mathrm{d}x &= \int_U \xi u D^\beta\left(D^\gamma \varphi\right)\,\mathrm{d}x \\ &= (-1)^{|\beta|}\int_U \sum_{\sigma\le\beta}\binom{\beta}{\sigma} D^\sigma \xi D^{\beta-\sigma} u D^\gamma \varphi\,\mathrm{d}x \\ &\overset{\text{Ind Hyp.}}{=} (-1)^{|\beta|+|\gamma|}\int_U \sum_{\sigma\le\beta}\binom{\beta}{\sigma} D^\gamma\left(D^\sigma \xi D^{\beta-\sigma} u\right)\varphi\,\mathrm{d}x \\ &\overset{\text{Ind Hyp.}}{=} (-1)^{|\alpha|}\int_U \sum_{\sigma\le\beta}\binom{\beta}{\sigma}\left(D^{\sigma+\gamma}\xi D^{\alpha-(\sigma+\gamma)} u + D^\sigma \xi D^{\alpha-\sigma} u\right)\varphi\,\mathrm{d}x \\ &= (-1)^{|\alpha|}\int_U \left(\sum_{\sigma\le\alpha}\binom{\alpha}{\sigma} D^\sigma \xi D^{\alpha-\sigma} u\right)\varphi\,\mathrm{d}x,\end{aligned}$$

   where for the last equality we have used the fact that

   $$\binom{\beta}{\sigma-\gamma} + \binom{\beta}{\sigma} = \binom{\alpha}{\sigma}.$$

□

4.4 Approximations

To better understand the properties of Sobolev spaces, it will be useful to develop systematic procedures to approximate Sobolev functions.

Theorem 4.4.1. *Let $u \in \mathrm{W}^{k,p}(U)$, for some $p \in [1,\infty)$, and set*

$$u_\epsilon := \eta_\epsilon \star u$$

in

$$U_\epsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \epsilon\},$$

and where η_ϵ is as given by (3.1.4) for the standard mollifier (3.1.3). Then the following hold.

1. *$u_\epsilon \in \mathcal{C}^\infty(U_\epsilon)$ for all $\epsilon > 0$.*
2. *If $V \Subset U$, then $u_\epsilon \to u$ in $\mathrm{W}^{k,p}(V)$ as $\epsilon \searrow 0$.*

Proof.

1. Fix $x \in U_\epsilon$, $i \in \{1, \dots, d\}$ and take h sufficiently small such that $x + he_i \in U_\epsilon$. Then
$$\frac{u_\epsilon(x+he_i) - u_\epsilon(x)}{h} = \frac{1}{\epsilon^d h} \int_{y \in U} \left(\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right) u(y)\,\mathrm{d}y.$$
Note
$$\frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right) \to \frac{\partial_{x_i}\eta\left(\frac{x-y}{\epsilon}\right)}{\epsilon}$$
uniformly in U. Thus,
$$\partial_{x_i} u_\epsilon(x) = \int_U \partial_{x_i}\eta_\epsilon(x-y)u(y)\,\mathrm{d}y.$$
Similarly, we show that
$$(D^\alpha u_\epsilon)(x) = \int_U D^\alpha \eta_\epsilon(x-y)u(y)\,\mathrm{d}y$$
for all $\alpha \in \mathbb{N}^d$.
2. Take $x \in U_\epsilon$, and note that
$$\begin{aligned} D^\alpha u_\epsilon(x) &= D^\alpha \int_U \eta_\epsilon(x-y)u(y)\,\mathrm{d}y \\ &= \int_U D^\alpha_x \eta_\epsilon(x-y)u(y)\,\mathrm{d}y \\ &= (-1)^{|\alpha|}\int_U D^\alpha_y \eta_\epsilon(x-y)u(y)\,\mathrm{d}y \\ &\overset{(1)}{=} (-1)^{2|\alpha|}\int_U \eta_\epsilon(x-y)D^\alpha u(y)\,\mathrm{d}y \end{aligned}$$
where (1) follows from the Definition 3.1.3, and the fact that η_ϵ is smooth. Hence,
$$D^\alpha u_\epsilon(x) = (\eta_\epsilon \star D^\alpha u)(x).$$
Now take $V \Subset U$, then $D^\alpha u_\epsilon \xrightarrow{\epsilon \searrow 0} D^\alpha u$ in $L^p(V)$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \le k$, by statement 2 of Lemma 3.1.10. Hence,
$$\begin{aligned} \|u_\epsilon - u\|^p_{\mathrm{W}^{k,p}(V)} &= \sum_{|\alpha| \le k} \int_V |D^\alpha u_\epsilon - D^\alpha u|^p\,\mathrm{d}x \\ &= \sum_{|\alpha|\le k} \|D^\alpha u_\epsilon - D^\alpha u\|^p_{L^p(V)} \\ &\xrightarrow{\epsilon\searrow 0} 0. \end{aligned}$$

$\square$

Remark 4.4.2. *Note that from the proof of statement 2 of Theorem 4.4.1 we deduce the more general result that for φ smooth and u weakly differentiable,*

$$(\varphi \star u)_{x_i} = \varphi_{x_i} \star u = \varphi \star u_{x_i}.$$

Theorem 4.4.3. *Let $U \subseteq \mathbb{R}^d$ be open and bounded and let $u \in \mathrm{W}^{k,p}(U)$ for $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then there exists $(u_m)_{m\in\mathbb{N}} \subseteq \mathcal{C}^\infty(U) \cap \mathrm{W}^{k,p}(U)$ such that $u_m \overset{m\to\infty}{\longrightarrow} u$ in $\mathrm{W}^{k,p}(U)$.*

Proof. Note that $U = \bigcup_{i=1}^\infty U_i$, where

$$U_i := \left\{ x \in U : \operatorname{dist}(x, \partial U) > \frac{1}{i} \right\}.$$

Let $V_i := U_{i+3} \setminus \bar{U}_{i+1}$ and choose $V_0 \Subset U$ open such that

$$U = \bigcup_{i\geq 0} V_i.$$

Now let $(\xi_i)_{i\in\mathbb{N}}$ be such that $\xi_i \in \mathcal{C}_c^\infty(V_i)$ and $\sum_{i\geq 1} \xi_i = 1$ on U. Such a collection $(\xi_i)_{i\in\mathbb{N}}$ is referred to as a partition of unity subordinate to the cover $(V_i)_{i\in\mathbb{N}}$. Note that on U the sum $\sum_{i\geq 1} \xi_i$ is always finite as each ξ_i has compact support. Then for $u \in \mathrm{W}^{k,p}(U)$ we have $\xi_i u \in W^{k,p}(U)$, by statement 4. of Theorem 4.3.1 and $\operatorname{supp}(\xi_i u) \subseteq V_i$. Fix $\delta > 0$, then for all i choose ϵ_i sufficiently small such that $u_i := \eta_{\epsilon_i} \star (\xi_i u)$ satisfies $\operatorname{supp}(u_i) \subseteq W_i$, where $W_i := U_{i+4} \setminus \bar{U}_i \Supset V_i$ and

$$\|u_i - \xi_i u\|_{\mathrm{W}^{k,p}(U)} = \|u_i - \xi_i u\|_{\mathrm{W}^{k,p}(W_i)} \leq \frac{\delta}{2^{i+1}}.$$

This can be done since $u_i \to \xi_i u$. Now write

$$v = \sum_{i\geq 0} u_i.$$

Observe that for $V \Subset U$, it follows for sufficiently large i that $\operatorname{supp}(u_i) \subseteq W_i$ and $W_i \cap V = \emptyset$. So on V the function v is the sum of finitely many smooth functions, meaning $v \in \mathcal{C}^\infty(V)$. In particular, as $u = \sum_{i\geq 0} \xi_i u$, for each $V \Subset U$ we have

$$\begin{aligned}
\|v - u\|_{\mathrm{W}^{k,p}(V)} &\leq \sum_{i\geq 0} \|u_i - \xi_i u\|_{\mathrm{W}^{k,p}(U)} \\
&\leq \delta \sum_{i\geq 0} \frac{1}{2^{i+1}} \\
&= \delta,
\end{aligned}$$

which is a bound independent on V. Therefore, taking the supremum over $V \Subset U$ we conclude that

$$\|v - u\|_{\mathrm{W}^{k,p}(U)} \leq \delta.$$

□

Remark 4.4.4. *Theorem 4.4.3 provides a global approximation of Sobolev functions using smooth functions, whereas, Theorem 4.4.1 only provides local approximations on compact subsets. Note that Theorem 4.4.3 and Theorem 4.4.1 only consider approximating functions defined on U. Under some conditions on the boundary of U, it is viable to approximate Sobolev functions with smooth functions that are defined up to the boundary of U.*

Definition 4.4.5. *Let $U \subseteq \mathbb{R}^d$ be open and bounded. Then ∂U is $\mathcal{C}^{k,\delta}$ if for all $p \in \partial U$ there exists an $r > 0$ and $\gamma \in \mathcal{C}^{k,\delta}\left(\mathbb{R}^{d-1}\right)$ such that, after possibly relabelling the axes, we have*

$$U \cap B_r(p) = \{(x_1, \ldots, x_{d-1}, x_d) := (x', x_d) \in B_r(p) : x_d > \gamma(x')\}.$$

Remark 4.4.6. *Intuitively, a $\mathcal{C}^{k,\delta}$ boundary locally is the graph of a $\mathcal{C}^{k,\delta}$ function.*

Lemma 4.4.7. *Let $p \in [1, \infty)$ and $g \in L^p\left(\mathbb{R}^d\right)$. Let $(z_j)_{j \in \mathbb{N}} \subseteq \mathbb{R}^d$ be a sequence such that $z_j \to 0$ as $j \to \infty$. Then*

$$\left\|\tau_{z_j} g - g\right\|_{L^p(\mathbb{R}^d)} \overset{j\to\infty}{\longrightarrow} 0,$$

where τ is the translation operator of statement 1 Definition 3.2.10.

Proof. Step 1: Let $g = \mathbf{1}_Q$ where $Q = (a_1, b_1) \times \cdots \times (a_d, b_d)$ with $I_m := b_m - a_m$ for $m = 1, \ldots, d$.
Observe that

$$\left\|\tau_{z_j} g - g\right\|_{L^p(\mathbb{R}^d)} \leq 2d|z_j| \max_{m=1,\ldots,d} (I_m)^{d-1}.$$

Hence, $\lim_{j\to\infty} \left\|\tau_{z_j} g - g\right\|_{L^p(\mathbb{R}^d)} = 0$.
Step 2: Let $g = \mathbf{1}_A$, where A is a measurable set of finite measure.
Fix $\epsilon > 0$. By the regularity of the Lebesgue measure, there exists a compact set $K \subseteq A$ and an open set $U \supseteq A$ such that the measure of $U \setminus K$ is less than ϵ. Since U is open we can write it as a collection of boxes

$$U = \bigcup_{\alpha \in \mathcal{A}} Q_\alpha.$$

Since, K is compact it is covered by finitely many of the boxes, more specifically,

$$K \subseteq \bigcup_{i=1}^{N} Q_i =: B.$$

As $K \subseteq B \subseteq U$, we have that $A \triangle B = (A \setminus B) \cup (B \setminus A) \subseteq U \setminus K$. Thus, as $p < \infty$, we have

$$\|\mathbf{1}_A - \mathbf{1}_B\|_{L^p(\mathbb{R}^d)} = \|\mathbf{1}_{A \triangle B}\|_{L^p(\mathbb{R}^d)} < \epsilon.$$

By step 1, we know there exists a $J \in \mathbb{N}$ such that for all $j \geq J$ we have

$$\left\|\tau_{z_j} \mathbf{1}_B - \mathbf{1}_B\right\|_{L^p(\mathbb{R}^d)} < \epsilon.$$

Therefore,

$$\begin{aligned}
\left\|\tau_{z_j} \mathbf{1}_A - \mathbf{1}_A\right\|_{L^p(\mathbb{R}^d)} &\leq \left\|\tau_{z_j} \mathbf{1}_A - \tau_{z_j} \mathbf{1}_B\right\|_{L^p(\mathbb{R}^d)} + \left\|\tau_{z_j} \mathbf{1}_B - \mathbf{1}_B\right\|_{L^p(\mathbb{R}^d)} + \|\mathbf{1}_B - \mathbf{1}_A\|_{L^p(\mathbb{R}^d)} \\
&= 2\|\mathbf{1}_A - \mathbf{1}_B\|_{L^p(\mathbb{R}^d)} + \left\|\tau_{z_j} \mathbf{1}_B - \mathbf{1}_B\right\|_{L^p(\mathbb{R}^d)} \\
&< 3\epsilon
\end{aligned}$$

for all $j \geq J$. Therefore, $\lim_{j\to\infty} \left\|\tau_{z_j} g - g\right\|_{L^p(\mathbb{R}^d)} = 0$.
Step 3: Let $g = \sum_{i=1}^{N} g_i \mathbf{1}_{A_i}$ for $g_i \in \mathbb{C}$ and A_i measurable sets of finite measure.
Then

$$\left\|\tau_{z_j} g - g\right\|_{L^p(\mathbb{R}^d)} \leq \sum_{i=1}^{N} |g_i| \left\|\tau_{z_i} \mathbf{1}_{A_i} - \mathbf{1}_{A_i}\right\|_{L^p(\mathbb{R}^d)}$$

and so by step 2 we have that $\lim_{j\to\infty} \left\|\tau_{z_j} g - g\right\|_{L^p(\mathbb{R}^d)} = 0$.
Step 4: Let $g \in L^p\left(\mathbb{R}^d\right)$.
Fix $\epsilon > 0$. Recall, that there exists a simple function $\bar{g}$ such that

$$\|\bar{g} - g\|_{L^p(\mathbb{R}^d)} < \epsilon.$$

By step 3 there exists a J such that

$$\left\|\tau_{z_j}\bar{g} - \bar{g}\right\|_{L^p(\mathbb{R}^d)} < \epsilon$$

for all $j \geq J$. Therefore,

$$\begin{aligned}\left\|\tau_{z_j} g - g\right\|_{L^p(\mathbb{R}^d)} &\leq \left\|\tau_{z_j} g - \tau_{z_j}\bar{g}\right\|_{L^p(\mathbb{R}^d)} + \left\|\tau_{z_j}\bar{g} - \bar{g}\right\|_{L^p(\mathbb{R}^d)} + \|\bar{g} - g\|_{L^p(\mathbb{R}^d)} \\ &= 2\|\bar{g} - g\|_{L^p(\mathbb{R}^d)} + \left\|\tau_{z_j}\bar{g} - \bar{g}\right\|_{L^p(\mathbb{R}^d)} \\ &< 3\epsilon.\end{aligned}$$

Therefore, $\lim_{j\to\infty} \left\|\tau_{z_j} g - g\right\|_{L^p(\mathbb{R}^d)} = 0$. □

Theorem 4.4.8. *Let $U \subseteq \mathbb{R}^d$ be open and bounded, and suppose that ∂U is $\mathcal{C}^{0,1}$, or in other words ∂U is Lipschitz. Let $u \in \mathrm{W}^{k,p}(U)$ for some $p \in [1,\infty)$. Then there exists $(u_m)_{m\in\mathbb{N}} \subseteq \mathcal{C}^\infty\left(\bar{U}\right)$ such that $u_m \to u$ in $\mathrm{W}^{k,p}(U)$.*

Proof. Step 1: Approximate locally around each point on the boundary.
Fix $x_0 \in \partial U$. Then, as ∂U is a Lipschitz boundary, there exists an $r > 0$ such that for some $\gamma \in \mathcal{C}^{0,1}\left(\mathbb{R}^{d-1}\right)$ we have

$$U \cap B_r(x_0) = \left\{(x', x_d) \in B_r(x_0) : x_d \geq \gamma(x')\right\}.$$

Let $V := U \cap B_{\frac{r}{2}}(x_0)$. For $x \in V$ and $\lambda, \epsilon > 0$ consider the shifted point $x^\epsilon := x + \lambda\epsilon e_d$. Fix $\lambda > 0$ and $\epsilon > 0$ such that $B_\epsilon(x^\epsilon)$ lies in $U \cap B_r(x_0)$ for all $x \in V$. Now let $u_\epsilon(x) := u(x^\epsilon)$ for $x \in V$. That is, u_ϵ is a translation of u by $\lambda\epsilon$ in the e_d-direction. Moreover, set

$$v_{\epsilon,\tilde{\epsilon}} := \eta_{\tilde{\epsilon}} \star u_\epsilon$$

for $0 \leq \tilde{\epsilon} < \epsilon$. Note that $v_{\epsilon,\tilde{\epsilon}} \in \mathcal{C}^\infty\left(\bar{V}\right)$.

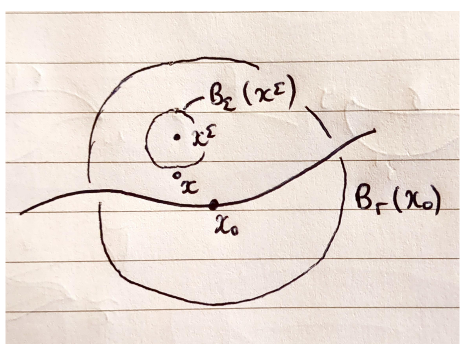


Figure 4.4.1: Moving away from the boundary, allowing us to mollify.

Fix $\delta > 0$. Note that

$$\begin{aligned}\|v_{\epsilon,\tilde{\epsilon}} - u\|_{\mathrm{W}^{k,p}(V)} &= \|v_{\epsilon,\tilde{\epsilon}} - u_\epsilon + u_\epsilon - u\|_{\mathrm{W}^{k,p}(V)} \\ &\leq \underbrace{\|v_{\epsilon,\tilde{\epsilon}} - u_\epsilon\|_{W^{k,p}(V)}}_{\mathrm{I}} + \underbrace{\|u_\epsilon - u\|_{\mathrm{W}^{k,p}(V)}}_{\mathrm{II}}.\end{aligned}$$

For II, observe that u_ϵ is a translation of u in the e_d-direction, which from Lemma 4.4.7 we know is a continuous function in L^p norm when $p < \infty$. In particular, we can choose $\epsilon > 0$ such that

$$\|u_\epsilon - u\|_{\mathrm{W}^{k,p}(V)} \leq \frac{\delta}{2}.$$

For I, with $\epsilon > 0$ sufficiently small for II, by statement 1 of Lemma 3.1.10 we can choose $0 < \tilde{\epsilon} < \epsilon$ such that

$$\|v_{\epsilon,\tilde{\epsilon}} - u_\epsilon\|_{\mathrm{W}^{k,p}(V)} \leq \frac{\delta}{2}.$$

Therefore,

$$\|v_{\epsilon,\tilde{\epsilon}} - u\|_{\mathrm{W}^{k,p}(V)} \leq \delta.$$

Note that the sets V for all $x_0 \in \partial U$ cover ∂U. Thus, since ∂U is compact, there exists $\left(x_0^i\right)_{i=1}^N \subseteq \partial U$, radii $(r_i)_{i=1}^N \subseteq \mathbb{R}_{>0}$ and sets $V_i := U \cap B_{\frac{r_i}{2}}\left(x_0^i\right)$ such that

$$\partial U \subseteq \bigcup_{i=1}^N B_{\frac{r_i}{2}}\left(x_0^i\right).$$

Moreover, we have the associated functions $v_i \in \mathcal{C}\left(\bar{V}_i\right)$ that satisfy

$$\|v_i - u\|_{\mathrm{W}^{k,p}(V_i)} \leq \delta \tag{4.4.1}$$

for each $i = 1, \ldots, N$.

Step 2: Use Theorem 4.4.3 to get an approximation of the interior.

Let $V_0 \Subset U$ be an open set such that

$$U \subseteq \bigcup_{i=0}^N V_i.$$

Then by Theorem 4.4.3, there exists a $v_0 \in \mathcal{C}\left(\bar{V}_0\right)$ such that

$$\|v_0 - u\|_{\mathrm{W}^{k,p}(V_0)} \leq \delta. \tag{4.4.2}$$

Step 3: Combine these approximations using a partition of unity.

Let $(\xi_i)_{i=0}^N$ be a partition of unity subordinate to the cover

$$\left\{V_0, B_{\frac{r_1}{2}}\left(x_0^1\right), \ldots, B_{\frac{r_n}{2}}\left(x_0^N\right)\right\}.$$

Let $\tilde{v}_\delta := \sum_{i=0}^N \xi_i v_i$, such that $v \in \mathcal{C}^\infty\left(\bar{U}\right)$. Furthermore, note that for $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ we have

$$\begin{aligned}
\|D^\alpha \tilde{v}_\delta - D^\alpha u\|_{L^p(U)} &\overset{(1)}{=} \left\| D^\alpha\left(\sum_{i=0}^N \xi_i v_i\right) - D^\alpha\left(\sum_{i=0}^N \xi_i u\right)\right\|_{L^p(U)} \\
&\leq C_k \sum_{i=0}^N \|D^\alpha(\xi_i v_i) - D^\alpha(\xi_i u)\|_{L^p(V_i)} \\
&\leq C_k \sum_{i=0}^N \|v_i - u\|_{\mathrm{W}^{k,p}(V_i)} \\
&\overset{(4.4.1)(4.4.2)}{\leq} C_k \delta(1+N) \\
&\overset{\delta \searrow 0}{\longrightarrow} 0.
\end{aligned}$$

where (1) follows as $\sum_{i\geq 0} \xi_i = 1$ on U, and C_k is some constant. Thus, the sequence $\left(\tilde{v}_{\frac{1}{m}}\right) \subseteq \mathcal{C}^\infty\left(\bar{U}\right)$ converges to u in $\mathrm{W}^{k,p}(U)$. $\square$

Proposition 4.4.9. *Let $u \in \mathrm{W}^{k,p}\left(\mathbb{R}^d\right)$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ such that $u_n \to u$ in $\mathrm{W}^{k,p}\left(\mathbb{R}^d\right)$.*

Proof. Let $\varphi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ be such that $0 \le \varphi(t) \le 1$ for all $t \in \mathbb{R}^d$ with $\varphi(t) = 1$ for $t \in B_1(0)$ and $\varphi(t) = 0$ for $t \in \mathbb{R}^d \setminus B_2(0)$. Let $\varphi_r(t) := \varphi\left(\frac{t}{r}\right)$, so that $\varphi_r(t) = 1$ for $t \in B_r(0)$ and $\operatorname{supp}(\varphi_r) \subseteq B_{2r}(0)$. From statement 4 of Theorem 4.3.1 we have $\varphi_r u \in \mathrm{W}^{k,p}\left(\mathbb{R}^d\right)$. For $\alpha \in \mathbb{N}^d$ with $|\alpha| \le k$ note that

$$D^\alpha\left(\varphi_r u - u\right) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left(D^\beta \varphi_r\right)\left(D^{\alpha-\beta} u\right) - D^\alpha u.$$

Observe that this implies that $D^\alpha\left(\varphi_r u - u\right)$ is supported on $\mathbb{R}^d \setminus B_r(0)$, thus

$$\|\varphi_r u - u\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)} = \|\varphi_r u - u\|_{\mathrm{W}^{k,p}(\mathbb{R}^d \setminus B_r(0))} \le \|u\|_{\mathrm{W}^{k,p}(\mathbb{R}^d \setminus B_r(0))},$$

where the last inequality follows as $0 \le \varphi(t) \le 1$. As $u \in \mathrm{W}^{k,p}\left(\mathbb{R}^d\right)$ it follows that we can choose an r_m such that

$$\|\varphi_r u - u\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)} \le \frac{\epsilon}{2}.$$

Since $\varphi_{r_m} u$ has compact support it can be approximated by a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$. In particular, there exists an $n \in \mathbb{N}$ such that

$$\|\varphi_{r_m} u - u_m\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)} \le \frac{\epsilon}{2}.$$

Therefore, by the triangle inequality it follows that

$$\|u_m - u\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)} \le \epsilon.$$

□

Exercise 4.4.10. *Show that $u \in L^2\left(\mathbb{R}^d\right)$ belongs to $\mathrm{H}^k\left(\mathbb{R}^d\right)$ if and only if*

$$\left(1 + |y|^k\right)\hat{u} \in L^2\left(\mathbb{R}^d\right),$$

where $\hat{u}$ denotes the Fourier transform of u.

4.5 Extensions

Previously, we investigated the approximation of Sobolev functions. Now we would like to extend a given Sobolev function on some space U to a larger space, such as $\mathbb{R}^d$. To do so it will be necessary to work with the boundary ∂U. Suppose that ∂U is a $\mathcal{C}^{1,0}$-boundary. That is, for all $q \in \partial U$ there exists $r > 0$ and $\gamma \in \mathcal{C}^{1,0}\left(\mathbb{R}^{d-1}\right)$ such that

$$U \cap B_r(q) = \left\{(x', x_d) \in B_r(q) : x_d > \gamma\left(x'\right)\right\}.$$

Consequently, we can straighten out the boundary. More specifically, let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be given by $\Phi(x) = y$ where

$$y_i = \begin{cases} x_i & i = 1, \dots, d-1 \\ x_d - \gamma(x_1, \dots, x_{d-1}) & i = d. \end{cases}$$

Then, $\partial U \mapsto \{y_d = 0\}$ under Φ. The inverse of Φ is $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ where $\Psi(y) = x$ is given by

$$x_i = \begin{cases} y_i & i = 1, \dots, d-1 \\ y_d + \gamma(y_1, \dots, y_{d-1}) & i = d. \end{cases}$$

In particular, $\Phi \circ \Psi = \Psi \circ \Phi = \mathrm{id}$ and

$$\Phi\left(U \cap B_r(q)\right) \subseteq \{y_d > 0\}.$$

Note that Φ and Ψ are $\mathcal{C}^1$-functions and as $\det(D\Phi) = \det(D\Psi) = 1$ the function Φ is a $\mathcal{C}^1$-diffeomorphism.

Exercise 4.5.1. *Let $U \subseteq \mathbb{R}^d$ with ∂U a $\mathcal{C}^1$-boundary given by Φ. Then for $u \in \mathrm{W}^{1,p}(U) \cap \mathcal{C}^1\left(\bar{U}\right)$ we have*

$$\|u \circ \Phi\|_{\mathrm{W}^{1,p}(\Psi(U))} \leq c\|u\|_{\mathrm{W}^{1,p}(U)},$$

where $\Psi := \Phi^{-1}$ and c is a constant independent of u.

Theorem 4.5.2. *Suppose $U \subseteq \mathbb{R}^d$ is open and bounded, with ∂U a $\mathcal{C}^1$-boundary. Then for $U \Subset V$ and $p \in [1, \infty]$ there exists a bounded linear operator $E : \mathrm{W}^{1,p}(U) \to \mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$ such that for all $u \in \mathrm{W}^{1,p}(U)$ the following hold.*

1. $E(u)|_U = u$ *almost everywhere.*
2. $\operatorname{supp}(E(u)) \subseteq V$.
3. $\|E(u)\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \leq c\|u\|_{\mathrm{W}^{1,p}(U)}$ *where* $c = c(U, V, p)$.

Proof. Fix $x_0 \in \partial U$.
Step 1: Consider the case when ∂U is flat near x_0 and $u \in \mathcal{C}^1\left(\bar{U}\right)$.
As ∂U is flat near x_0, it lies in the plane $\{x_d = 0\}$. Thus we may assume that there exists a $r > 0$ such that

$$B_+ := B_r(x_0) \cap \{x_d \geq 0\} \subseteq \bar{U}$$

and

$$B_- := B_r(x_0) \cap \{x_d \leq 0\} \subseteq \mathbb{R}^d \setminus U.$$

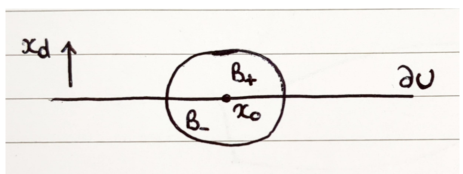


Figure 4.5.1

Consider

$$\bar{u}(x) := \begin{cases} u(x) & x \in B_+ \\ -3u\left(x', -x_d\right) + 4u\left(x', -\frac{x_d}{2}\right) & x \in B_-. \end{cases}$$

Note that $\bar{u}$ is continuous on $\{x_d = 0\}$, namely $\bar{u}(x) = u\left(x', 0\right)$ on the boundary, thus $\bar{u} \in \mathcal{C}^0\left(B_r(x_0)\right)$. Moreover, for $1 \leq k \leq d-1$ we have

$$\bar{u}_{x_k}(x) = \begin{cases} u_{x_k}(x) & x \in B_+ \\ -3u_{x_k}\left(x' - x_d\right) + 4u_{x_k}\left(x', -\frac{x_d}{2}\right) & x \in B_-, \end{cases}$$

and for $k = d$ we have

$$\bar{u}_{x_d}(x) = \begin{cases} u_{x_d}(x) & x \in B_+ \\ 3u_{x_d}\left(x', -x_d\right) - 2u\left(x', -\frac{x_d}{2}\right) & x \in B_-, \end{cases}$$

which are continuous on $\{x_d = 0\}$ and so $\bar{u} \in \mathcal{C}^1\left(B_r(x_0)\right)$. Using these computations it follows that

$$\|\bar{u}\|_{\mathrm{W}^{1,p}(B_r(x_0))} \leq c\|u\|_{\mathrm{W}^{1,p}(B_+)}$$

for some $c \geq 0$ independent of u. Hence, in this case, one can take $E(u) := \bar{u}$.
Step 2: Consider the case when ∂U is a $\mathcal{C}^1$-boundary near x_0 and $u \in \mathcal{C}^1\left(\bar{U}\right)$.
Since, ∂U is a $\mathcal{C}^1$-boundary there exists an open neighbourhood V of x_0 such that boundary ∂U is the graph

of a function Φ. In particular, the map Φ is an open map as it has an inverse Ψ, and so $\Phi(V)$ is an open neighbourhood of $y_0 = \Phi(x_0)$. Hence, there exists an $r > 0$ such that $B_r(y_0) \subseteq \Phi(V)$. As Ψ is also open, the set $\Psi(B_r(y_0))$ is an open neighbourhood of x_0. Let $W := \Psi(B_r(y_0))$. Note that $\Phi(W) = B_r(y_0)$ and

$$\Phi(U \cap W) = B_r(y_0) \cap \{y_d > 0\}.$$

Let $y = \Phi(x)$ and $x = \Psi(y)$. Consider $\tilde{u}(y) := u(\Psi(y))$, and the sets B_+ and B_- for y_0 as constructed in step 1. As $\tilde{u} \in \mathcal{C}^1(B_+)$ we know from by step 1 that there exists an extension $\tilde{\tilde{u}} \in \mathcal{C}^1(B_r(y_0))$ such that $\tilde{\tilde{u}}$ extends $\tilde{u}$ and

$$\|\tilde{\tilde{u}}\|_{\mathrm{W}^{1,p}(B_r(y_0))} \leq c\,\|\tilde{u}\|_{\mathrm{W}^{1,p}(B_+)}. \tag{4.5.1}$$

Let

$$\bar{u}(x) := \tilde{\tilde{u}}(\Phi(x)).$$

Note that $\bar{u} \in \mathcal{C}^1(W)$ and extends u as $\Phi \circ \Psi = \mathrm{id}$. Observe that

$$\begin{aligned}
\|\bar{u}\|_{\mathrm{W}^{1,p}(W)} &= \|\tilde{\tilde{u}} \circ \Phi\|_{\mathrm{W}^{1,p}(\Psi(B_r(y_0)))} \\
&\overset{\text{Ex. 4.5.1}}{\leq} c\,\|\tilde{\tilde{u}}\|_{\mathrm{W}^{1,p}(B_r(y_0))} \\
&\overset{(4.5.1)}{\leq} c'\,\|\tilde{u}\|_{\mathrm{W}^{1,p}(B_+)} \\
&= c'\,\|u \circ \Phi\|_{\mathrm{W}^{1,p}(B_+)} \\
&\overset{\text{Ex. 4.5.1}}{\leq} c''\|u\|_{\mathrm{W}^{1,p}(\Psi^{-1}(B_+))} \\
&= c''\|u\|_{\mathrm{W}^{1,p}(U)}
\end{aligned}$$

where c is some constant independent of u. Thus we have established local extensions at each point of the boundary, therefore, we can use the compactness of ∂U to determine finitely many points $x_0^i \in \partial U$ with corresponding sets W_i such that $\bar{u}_i \in \mathcal{C}^1(W_i)$ extends u and $\partial U \subseteq \bigcup_{i=1}^N W_i$. Moreover, there exists $W_0 \Subset U$ such that

$$U \subseteq \bigcup_{i=0}^{N} W_i.$$

Take $(\xi_i)_{i=0}^N$ to be a partition of unity subordinate to $(W_i)_{i=0}^N$ and let $\bar{u} = \sum_{i=0}^N \xi_i \bar{u}_i$, where $\bar{u}_0 = u$. Observe that on U we have $\bar{u} = u$ as $u_i = 0$ on U for all i and $\sum_{i=0}^N \xi_i = 1$. Moreover, $\bar{u} \in \mathcal{C}^1(\mathbb{R}^d)$ since the ξ_i vanish outside a compact set. Furthermore,

$$\|\bar{u}\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \leq c\|u\|_{\mathrm{W}^{1,p}(U)},$$

and so we can conclude by letting $E(u) := \bar{u}$.

Step 3: Consider the case when ∂U is a $\mathcal{C}^1$ boundary and $u \in \mathrm{W}^{1,p}(U)$.

Take $u \in \mathrm{W}^{1,p}(U)$ and find $(u_m)_{m\in\mathbb{N}} \subseteq \mathcal{C}^\infty(\bar{U})$ converging to u in $\mathrm{W}^{1,p}(U)$. Using step 2 we can consider the sequence $(E(u_m))_{m\in\mathbb{N}} \subseteq \mathrm{W}^{1,p}(\mathbb{R}^d)$. By linearity

$$\begin{aligned}
\|E(u_m) - E(u_k)\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} &= \|E(u_m - u_k)\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \\
&\leq c\|u_m - u_k\|_{\mathrm{W}^{1,p}(U)},
\end{aligned}$$

where the inequality follows from step 2 which can be applied as $u_m - u_k \in \mathcal{C}^\infty(\bar{U})$. As $(u_m)_{m\in\mathbb{N}}$ is Cauchy in $\mathrm{W}^{1,p}(U)$ it follows that the sequence $(E(u_m))_{m\in\mathbb{N}} \subseteq \mathrm{W}^{1,p}(\mathbb{R}^d)$ is Cauchy and thus convergent as $\mathrm{W}^{1,p}(\mathbb{R}^d)$ is complete. Hence, one can let

$$E(u) := \lim_{m\to\infty} E(u_m),$$

which is well-defined as limits are unique and so the limit is independent of the exact sequence chosen in $\mathcal{C}^\infty(\bar{U})$ which converges to u. Using step 2 all the requirements of E are satisfied and so this completes the proof. $\square$

Remark 4.5.3.

1. *Theorem 4.5.2 shows the existence of an operator that extends u to a larger region. Hence, E is referred to as an extension operator.*

2. *An analogous result holds for extending functions on $\mathrm{W}^{k,p}(U)$. To prove this generalised result, in step 1 one would have to extend u on $x_d < 0$ with*

$$\sum_{i=1}^{k} c_i u\left(x', -\frac{x_d}{2}\right)$$

where the c_i are such that $\sum_{i=1}^{k} c_i \left(-\frac{1}{i}\right)^m = 1$ for all $m = 0, 1, \ldots, k-1$.

4.6 Trace Operator

The restriction of a function $u \in \mathcal{C}\left(\bar{U}\right)$ to ∂U is well-defined as u has a pointwise construction. However, a function $u \in \mathrm{W}^{1,p}(U)$ is only defined almost surely, thus its restriction to ∂U has no meaning since ∂U is a set of measure zero. In particular, this means that there are no guarantees of continuity. This is an issue as the boundary values for elliptic partial differential equations are important as they influence the solvability of such equations. The trace operator resolves this issue.

Lemma 4.6.1 (Young's Inequality)**.** *If $p, q \in (1, \infty)$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for all $a, b > 0$.

Proof. The function $f(x) = e^x$ is convex, hence

$$\begin{aligned} ab &= f(\log(a) + \log(b)) \\ &= f\left(\frac{1}{p}\log\left(a^p\right) + \frac{1}{q}\log\left(b^q\right)\right) \\ &\leq \frac{1}{p} f\left(\log\left(a^p\right)\right) + \frac{1}{q} f\left(\log\left(b^q\right)\right) \\ &= \frac{1}{p} e^{\log(a^p)} + \frac{1}{q} e^{\log(b^q)} \\ &= \frac{a^p}{p} + \frac{b^q}{q}. \end{aligned}$$

□

Remark 4.6.2. *A convenient specification of Lemma 4.6.1 arises when setting $\tilde{a} = (\epsilon p)^{\frac{1}{p}} a$ and $\tilde{b} = (\epsilon p)^{-\frac{1}{p}} b$ for $\epsilon > 0$ to give*

$$ab = \tilde{a}\tilde{b} \leq \frac{\tilde{a}^p}{p} + \frac{\tilde{b}^q}{q} = \epsilon a^p + C(\epsilon) b^q,$$

where $C(\epsilon) = \frac{(\epsilon p)^{-\frac{q}{p}}}{q}$.

Theorem 4.6.3. *Let $p \in [1, \infty)$. Let $U \subseteq \mathbb{R}^d$ be an open and bounded set with ∂U a $\mathcal{C}^1$-boundary. Then there exists a bounded linear operator $T : \mathrm{W}^{1,p}(U) \to L^p(\partial U)$ such that the following hold.*

> 1. $Tu = u|_{\partial U}$ *if* $u \in \mathrm{W}^{1,p}(U) \cap \mathcal{C}\left(\bar{U}\right)$.
>
> 2. $\|Tu\|_{L^p(\partial U)} \leq c\|u\|_{\mathrm{W}^{1,p}(U)}$, *for* $u \in \mathrm{W}^{1,p}(U)$ *and* $c = c(p, U)$.

Proof. Fix $x_0 \in \partial U$.
Step 1: Consider the case when ∂U is flat near x_0 and $u \in \mathcal{C}^1\left(\bar{U}\right)$.
As ∂U is flat near x_0, it lies in the plane $\{x_d = 0\}$. Thus, we may assume that there exists a $r > 0$ such that

$$B_+ := B_r(x_0) \cap \{x_d \geq 0\} \subseteq \bar{U}$$

and

$$B_- := B_r(x_0) \cap \{x_d \leq 0\} \subseteq \mathbb{R}^d \setminus U.$$

Let $\hat{B} := B_{\frac{r}{2}}(x_0)$. Let $\xi \in \mathcal{C}_c^\infty(B)$ be such that $\xi \geq 0$ on B and $\xi \equiv 1$ on $\hat{B}$. Let $\Gamma := \partial U \cap \hat{B}$. Then with $x' = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1} = \{x_d = 0\}$ it follows that

$$\begin{aligned}
\int_\Gamma |u|^p \,\mathrm{d}x' &\leq \int_{\{x_d=0\}} \xi|u|^p \,\mathrm{d}x' \\
&\overset{(1)}{=} \int_{\{x_d=0\}} \int_0^\infty -\left(\xi|u|^p\right)_{x_d} \,\mathrm{d}x_d \,\mathrm{d}x' \\
&\overset{(2)}{=} -\int_{B_+} \left(\xi|u|^p\right)_{x_d} \,\mathrm{d}x \\
&= -\int_{B_+} |u|^p \xi_{x_d} + p|u|^{p-1}\mathrm{sgn}(u) u_{x_d} \xi \,\mathrm{d}x \\
&\overset{\text{Lem. 4.6.1}}{\leq} c_1 \int_{B+} |u|^p \,\mathrm{d}x + c_2 \int_{B_1} \left(|u|^{p-1}\right)^{\frac{p}{p-1}} + |u_{x_d}|^p \,\mathrm{d}x \\
&\leq c \int |u|^p + |Du|^p \,\mathrm{d}x. \qquad (4.6.1)
\end{aligned}$$

where (1) is an application of the fundamental theorem of calculus, and the fact that u vanishes at ∞. Similarly, (2) holds as ξ vanishes outside of B. Hence, $T(u) := u|_{\partial U}$ is bounded with

$$\|T(u)\|_{L^p(\partial U)} \leq c(p, U)\|u\|_{\mathrm{W}^{1,p}(U)}$$

for any $u \in \mathcal{C}^1\left(\bar{U}\right)$.
Step 2: Consider the case when ∂U is a $\mathcal{C}^1$-boundary and $u \in \mathcal{C}^1\left(\bar{U}\right)$.
As done in the proof of Theorem 4.5.2, we straighten out the boundary and the use (4.6.1) from step 1 to deduce that

$$\int_\Gamma |u|^p \,\mathrm{d}S \leq c \int_U |u|^p + |Du|^p \,\mathrm{d}x \qquad (4.6.2)$$

where $\Gamma \subseteq \partial U$ is open and contains x_0.
Step 3: Use the compactness of ∂U to find an estimate on the boundary.
Since ∂U is compact there exists finitely many points $\left(x_i^0\right)_{i=1}^N$ with corresponding open sets $(\Gamma_i)_{i=1}^N \subseteq \partial U$ such that $\partial U = \bigcup_{i=1}^N \Gamma_i$ and

$$\|u\|_{L^p(\Gamma_i)} \leq c_i\|u\|_{\mathrm{W}^{1,p}(U)}$$

for $i = 1, \ldots, N$ by (4.6.2). Letting $T(u) := u|_{\partial U}$ we have

$$\|Tu\|_{L^p(\partial U)} \leq c\|u\|_{\mathrm{W}^{1,p}(U)},$$

where c is a constant independent of u.
Step 4: Consider $u \in \mathrm{W}^{1,p}(U)$.
There exists a sequence of function $(u_m)_{m\in\mathbb{N}} \subseteq \mathcal{C}^\infty\left(\bar{U}\right)$ converging to u in $\mathrm{W}^{1,p}(U)$. Using (4.6.2) from step 2 we have

$$\|Tu_m - Tu_l\|_{L^p(\partial U)} \leq c\|u_m - u_l\|_{\mathrm{W}^{1,p}(U)}.$$

Hence, $(Tu_m)_{m\in\mathbb{N}} \subseteq L^p(\partial U)$ is a Cauchy sequence, and thus we can let

$$T(u) := \lim_{m\to\infty} T(u_m),$$

where the limit is taken $L^p(\partial U)$. Observe that,

$$\begin{aligned}\|Tu\|_{L^p(\partial U)} &= \lim_{m\to\infty} \|Tu_m\|_{L^p(\partial u)} \\ &\leq \lim_{m\to\infty} c\|u_m\|_{\mathrm{W}^{1,p}(U)} \\ &= c\|u\|_{W^{1,p}(U)}.\end{aligned}$$

We note by (4.6.2) that $T(u)$ is independent of the chosen sequence of smooth functions.
Step 5: Consider $u \in \mathrm{W}^{1,p}(U) \cap \mathcal{C}\left(\bar{U}\right)$.
Using Theorem 4.4.8 one can choose a sequence converging to u in $\mathcal{C}\left(\bar{U}\right)$, thus $T(u) = u|_{\partial U}$. □

Remark 4.6.4.

1. *The operator T of Theorem 4.6.3 is referred to as the trace operator, with $T(u)$ referred to as the trace of u on ∂U. Similar trace operators exists for $u \in \mathrm{W}^{k,p}(U)$.*

2. *Note how statement 2 of Theorem 4.6.3 effectively says that T is continuous, since bounded operators are continuous. In particular, this means that for $u \in \mathrm{W}^{1,p}(U)$, as $\mathcal{C}\left(\bar{U}\right) \cap \mathrm{W}^{1,p}(U)$ is dense in $\mathrm{W}^{1,p}(U)$ we can first define $u|_{\partial U}$ for $u \in \mathcal{C}\left(\bar{U}\right) \cap \mathrm{W}^{1,p}(U)$ using statement 1 of Theorem 4.6.3. Then, we can uniquely extend u to the boundary by taking a convergent sequence in $\mathcal{C}\left(\bar{U}\right) \cap \mathrm{W}^{1,p}(U)$ and defining u on ∂U as the limit of T applied to the convergent sequence. We can do this uniquely as T is continuous by statement 2 of Theorem 4.6.3.*

Theorem 4.6.5. *Let $p \in [1, \infty)$. Let $U \subseteq \mathbb{R}^d$ be an open and bounded set with ∂U a $\mathcal{C}^1$-boundary. Then for $u \in \mathrm{W}^{1,p}(U)$ we have $u \in \mathrm{W}_0^{1,p}(U)$ if and only if $T(u) = 0$ on ∂U.*

Proof. ($\Rightarrow$). By construction there exists a sequence of function $(u_m)_{m\in\mathbb{N}} \subseteq \mathcal{C}_c^\infty(U)$ such that $u_m \to u$ in $\mathrm{W}^{1,p}(U)$. Note each u_m is compactly supported so that $T(u_m) = 0$ and so as $T : \mathrm{W}^{1,p}(U) \to L^p(\partial U)$ is a bounded linear operator, it follows that $Tu = 0$ on ∂U.
($\Leftarrow$). Using a partition of unity and flattening the boundary, we may assume without loss of generality that $u \in \mathrm{W}^{1,p}\left(\mathbb{R}_+^d\right)$ has compact support in $\bar{\mathbb{R}}_+^d$ and $T(u) = 0$ on $\partial\mathbb{R}_+^d = \mathbb{R}^{d-1}$. Consequently, there exists a sequence of functions $(u_m)_{m\in\mathbb{N}} \subseteq \mathcal{C}^1\left(\bar{\mathbb{R}}_+^d\right)$ that converges to u in $\mathrm{W}^{1,p}\left(\mathbb{R}_+^d\right)$ and $T(u_m) = u_m|_{\mathbb{R}^{d-1}} \to 0$ in $L^p\left(\mathbb{R}^{d-1}\right)$. Observe that if $x' \in \mathbb{R}^{d-1}$ and $x_d \geq 0$, then

$$|u_m(x', x_d)| \leq |u_m(x', 0)| + \int_0^{x_d} |u_{m,x_d}(x', t)| \,\mathrm{d}t,$$

thus

$$\int_{\mathbb{R}^{d-1}} |u_m(x', x_d)|^p \,\mathrm{d}x' \leq c\left(\int_{\mathbb{R}^{d-1}} |u_m(x', 0)|^p \,\mathrm{d}x' + x_d^{p-1}\int_0^{x_d}\int_{\mathbb{R}^{d-1}} |Du_m(x', t)|^p \,\mathrm{d}x'\,\mathrm{d}t\right).$$

Letting $m \to \infty$ we deduce that

$$\int_{\mathbb{R}^{d-1}} |u(x', x_d)|^p \,\mathrm{d}x' \leq cx_d^{p-1}\int_0^{x_d}\int_{\mathbb{R}^{d-1}} |Du|^p \,\mathrm{d}x'\,\mathrm{d}t \tag{4.6.3}$$

for almost every $x_d > 0$. Now consider $\xi \in \mathcal{C}^\infty(\mathbb{R}_+)$ such that $\xi \equiv 1$ on $[0, 1]$, $\xi \equiv 0$ on $\mathbb{R}_+ \setminus [0, 2]$ and $0 \leq \xi \leq 1$. Moreover, let

$$\xi_m(x) := \xi(mx_d)$$

for $x \in \mathbb{R}^d_+$ and

$$w_m := u(x)(1-\xi_m).$$

Then

$$w_{m,x_d} = u_{x_d}(1-\xi_m) - mu\xi'$$

and

$$D_{x'}w_m = D_{x'}u(1-\xi_m).$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^d_+} |Dw_m - Du|^p \,\mathrm{d}x &\leq c\int_{\mathbb{R}^d_+} |\xi_m|^p |Du|^p \,\mathrm{d}x + cm^p \int_0^{\frac{2}{m}} \int_{\mathbb{R}^{d-1}} |u|^p \,\mathrm{d}x' \,\mathrm{d}t \\ &=: A + B. \end{aligned} \tag{4.6.4}$$

Note that $A \to 0$ as $m \to \infty$ since $\xi_m \neq 0$ if and only if $0 \leq x_n \leq \frac{2}{m}$. Similarly,

$$\begin{aligned} B &\overset{(4.6.3)}{\leq} cm^p \left(\int_0^{\frac{2}{m}} t^{p-1} \,\mathrm{d}t \right) \left(\int_0^{\frac{2}{m}} \int_{\mathbb{R}^{d-1}} |Du|^p \,\mathrm{d}x' \,\mathrm{d}x_d \right) \\ &\leq c \int_0^{\frac{2}{m}} \int_{\mathbb{R}^{d-1}} |Du|^p \,\mathrm{d}x' \mathrm{d}x_d \\ &\overset{m\to\infty}{\longrightarrow} 0 \end{aligned}$$

Hence, from (4.6.4) it follows that $Dw_m \to Du$ in $L^p\left(\mathbb{R}^d_+\right)$. As we also have $w_m \to u$ in $L^p\left(\mathbb{R}^d_+\right)$ it follows that $w_m \to u$ in $\mathrm{W}^{1,p}\left(\mathbb{R}^d_+\right)$. Therefore, as $w_m = 0$ if $0 < x_d < \frac{1}{m}$, we can mollify the w_m to construct a sequence of functions $u_m \in \mathcal{C}^\infty_c\left(\mathbb{R}^d_+\right)$ such that $u_m \to u$ in $\mathrm{W}^{1,p}\left(\mathbb{R}^d_+\right)$. Hence, $u \in \mathrm{W}^{1,p}_0\left(\mathbb{R}^d_+\right)$. □

4.7 Sobolev Inequalities

It will be interesting now to understand how Sobolev spaces are embedded into one another. To do so we will develop Sobolev inequalities. In particular, we consider $u \in \mathrm{W}^{1,p}(U)$ and understand when it lies in other spaces.

Lemma 4.7.1. *Let $1 \leq p_1, \ldots, p_m \leq \infty$ be such that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$. Then with $u_k \in L^{p_k}(U)$ for $k = 1, \ldots, m$ we have*

$$\int_U |u_1 \ldots u_m| \,\mathrm{d}x \leq \prod_{k=1}^m \|u_k\|_{L^{p_k}(U)}.$$

Proof.

- For $m = 1$ we have $\frac{1}{p_1} = 1$, thus

$$\int_U |u_1| \,\mathrm{d}x = \|u_1\|_{L^{p_1}(U)}.$$

- Suppose the result holds for $m \geq 1$. Then for $1 \leq p_1, \ldots, p_m, p_{m+1} \leq \infty$ such that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{p_{m+1}} = 1$ consider $u_k \in L^{p_k}(U)$ for $k = 1, \ldots, m+1$. Let

$$\frac{1}{q} := \frac{1}{p_1} + \cdots + \frac{1}{p_m}$$

such that

$$\frac{1}{\left(\frac{p_1}{q}\right)} + \cdots + \frac{1}{\left(\frac{p_m}{q}\right)} = 1.$$

Then note that

$$
\begin{aligned}
\||u_k|^q\|_{L^{\frac{p_k}{q}}(U)} &= \left(\int_U \left(|u_k|^q\right)^{\frac{p_k}{q}} \, \mathrm{d}x\right)^{\frac{q}{p_k}} \\
&= \left(\int_U |u|^{p_k} \, \mathrm{d}x\right)^{\frac{q}{p_k}} \\
&= \|u_k\|_{L^{p_k}(U)}^q,
\end{aligned}
$$

which implies that $|u_k|^q \in L^{\frac{p_k}{q}}(U)$ for each $k = 1, \ldots, m$. Therefore, applying the inductive hypothesis we have

$$
\begin{aligned}
\int_U \left||u_1|^q \ldots |u_m|^q\right| \, \mathrm{d}x &\leq \prod_{k=1}^m \||u_k|^q\|_{L^{\frac{p_k}{q}}(U)} \\
&= \prod_{k=1}^m \|u_k\|_{L^{p_k}(U)}^q.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_U |u_1 \ldots u_m u_{m+1}| \, \mathrm{d}x &\overset{\text{Hölder's}}{\leq} \left(\int_U \left||u_1|^q \ldots |u_m|^q\right| \, \mathrm{d}x\right)^{\frac{1}{q}} \|u_{m+1}\|_{L^{p_{m+1}}(U)} \\
&\leq \left(\prod_{k=1}^m \|u_k\|_{L^{p_k}(U)}^q\right)^{\frac{1}{q}} \|u_{m+1}\|_{L^{p_{m+1}}(U)} \\
&= \left(\prod_{k=1}^m \|u_k\|_{L^{p_k}(U)}\right) \|u_{m+1}\|_{L^{p_{m+1}}(U)} \\
&= \prod_{k=1}^{m+1} \|u_k\|_{L^{p_k}(U)}.
\end{aligned}
$$

□

Lemma 4.7.2. *For $d \geq 2$ let $f_1, \ldots, f_d \in L^{d-1}\left(\mathbb{R}^{d-1}\right)$. Then set*

$$
f(x) := \prod_{i=1}^d f_i\left(\tilde{x}_i\right)
$$

where for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we let $\tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$. It follows that $f \in L^1\left(\mathbb{R}^d\right)$ with

$$
\|f\|_{L^1(\mathbb{R}^d)} \leq \prod_{i=1}^d \|f_i\|_{L^{d-1}(\mathbb{R}^{d-1})}.
$$

Proof. We proceed by induction on d.

- For $d = 2$ we have $f(x_1, x_2) = f_1(x_2) f_2(x_1)$. Thus,

$$
\begin{aligned}
\|f\|_{L^1(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |f(x_1, x_2)| \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\overset{\text{Fubini}}{=} \int_{\mathbb{R}} |f_1(x_2)| \, \mathrm{d}x_2 \int_{\mathbb{R}} |f_2(x_1)| \, \mathrm{d}x_1 \\
&= \|f_1\|_{L^1(\mathbb{R})} \|f_2\|_{L^1(\mathbb{R})}.
\end{aligned}
$$

- Suppose that the result holds for $d \geq 2$. For the case $d+1$ we have $f(x) = f_{d+1}(\tilde{x}_{d+1}) F(x)$ where $F(x) := f_1(\tilde{x}_1) \dots f_d(\tilde{x}_d)$. Observe that

$$\int_{x_1,\dots,x_d} |f(x, x_{d+1})| \, \mathrm{d}x_1 \dots \mathrm{d}x_d = \int_{x_1,\dots,x_d} |f_{d+1}(\tilde{x}_{d+1})| \, |F(x)| \, \mathrm{d}x_1 \dots \mathrm{d}x_d \\ \leq \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \|F\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)},$$

where the inequality comes from an application of Hölder's inequality with $p = d$ and $q = \frac{d}{d-1}$. Recall that

$$F(x, x_{d+1})^{\frac{d}{d-1}} = f_1(x, x_{d+1})^{\frac{d}{d-1}} \dots f_d(x, x_{d+1})^{\frac{d}{d-1}},$$

and so by applying the inductive assumption it follows that

$$\int_{x_1,\dots,x_d} |f(x, x_{d+1})| \, \mathrm{d}x_1 \dots \mathrm{d}x_d \leq \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \left(\prod_{i=1}^{d} \left\| f_i(\cdot, x_{d+1})^{\frac{d}{d-1}} \right\|_{L^{d-1}(\mathbb{R}^{d-1})} \right)^{\frac{d-1}{d}} \\ = \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{i=1}^{d} \|f_i(\cdot, x_{d+1})\|_{L^d(\mathbb{R}^{d-1})}.$$

Now integrating over x_{d+1} it follows that

$$\|f\|_{L^1(\mathbb{R}^{d+1})} \leq \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \int_{\mathbb{R}} \prod_{i=1}^{d} \|f_i(\cdot, x_{d+1})\|_{L^d(\mathbb{R}^{d-1})} \, \mathrm{d}x_{d+1} \\ \overset{\text{Lem4.7.1}}{\leq} \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{i=1}^{d} \left(\int_{\mathbb{R}} \|f_i(\cdot, x_{d+1})\|^d_{L^d(\mathbb{R}^{d-1})} \, \mathrm{d}x_{d+1} \right)^{\frac{1}{d}} \\ = \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{i=1}^{d} \left(\int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} |f_i(x)|^d \mathrm{d}x \, \mathrm{d}x_{d+1} \right)^{\frac{1}{d}} \\ = \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{i=1}^{d} \|f_i\|_{L^d(\mathbb{R}^d)}.$$

□

4.7.1 Gagliardo-Nirenberg-Sobolev Inequality

Example 4.7.3. *For $p \in [1, d)$ there are specific values $q \in [1, \infty)$ such that an inequality of the form*

$$\|u\|_{L^q(\mathbb{R}^d)} \leq c \|Du\|_{L^p(\mathbb{R}^d)} \tag{4.7.1}$$

could hold for all $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and some constant c independent of u. Indeed, let $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ with $u \not\equiv 0$ and let

$$u_\lambda(x) := u(\lambda x)$$

for $\lambda > 0$. Then (4.7.1) *implies that*

$$\|u_\lambda\|_{L^q(\mathbb{R}^d)} \leq c \|Du_\lambda\|_{L^p(\mathbb{R}^d)}.$$

As

$$\int_{\mathbb{R}^d} |u_\lambda|^q \, \mathrm{d}x = \int_{\mathbb{R}^d} |u(\lambda x)|^q \, \mathrm{d}x = \frac{1}{\lambda^d} \int_{\mathbb{R}^d} |u(y)|^q \, \mathrm{d}y$$

and,

$$\int_{\mathbb{R}^d} |Du_\lambda|^p \, \mathrm{d}x = \frac{\lambda^p}{\lambda^d} \int_{\mathbb{R}^d} |Du(y)|^p \, \mathrm{d}y,$$

it follows that

$$\frac{1}{\lambda^{\frac{d}{q}}}\|u\|_{L^q(\mathbb{R}^d)} \leq c\frac{\lambda}{\lambda^{\frac{d}{p}}}\|Du\|_{L^p(\mathbb{R}^d)}.$$

Thus,

$$\|u\|_{L^q(\mathbb{R}^d)} \leq c\lambda^{1-\frac{d}{p}+\frac{d}{q}}\|Du\|_{L^p(\mathbb{R}^d)}.$$

Hence, if $1-\frac{d}{p}+\frac{d}{q}<0$, then by sending $\lambda\to\infty$ we deduce that $u\equiv 0$. Similarly, if $1-\frac{d}{p}+\frac{d}{p}>0$, then by sending $\lambda\to 0$ we deduce that $u\equiv 0$. These are contradictions since we assume $u\not\equiv 0$, therefore, it must be the case that $1-\frac{d}{p}+\frac{d}{q}=0$. In particular, if an inequality such as (4.7.1) exists, it must be the case that $q=\frac{dp}{d-p}$.

Definition 4.7.4. *For $p\in[1,d)$, the Sobolev conjugate of p is*

$$p^* := \frac{dp}{d-p}.$$

Remark 4.7.5. *Note that*

$$\frac{1}{p^*}=\frac{1}{p}-\frac{1}{d},$$

meaning $p^*>p$.

Theorem 4.7.6 (Gagliardo-Nirenberg-Sobolev Inequality)**.** *On $\mathbb{R}^d$, let $p\in[1,d)$. Then there exists a constant $c(p,d)$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq c\|Du\|_{L^p(\mathbb{R}^d)}$$

for all $u\in \mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$. That is,

$$\mathrm{W}^{1,p}\left(\mathbb{R}^d\right) \hookrightarrow L^{p^*}\left(\mathbb{R}^d\right).$$

Proof. Step 1: Consider the case when $p=1$.
Let $u\in\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$. Since u has compact support, by the fundamental theorem of calculus we have

$$u(x)=\int_{-\infty}^{x_i} u_{x_i}\left(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_d\right)\mathrm{d}y_i$$

for each $i=1,\dots,d$ and $x\in\mathbb{R}^d$. Hence,

$$|u(x)|\leq\int_{-\infty}^{\infty}|Du(x_1,\dots,y_i,\dots,x_d)|\,\mathrm{d}y_i =: f_i\left(\tilde{x}_i\right)$$

for $i=1,\dots,d$. Therefore,

$$|u(x)|^{\frac{d}{d-1}} \leq (\underbrace{|u|\dots|u|}_{d})^{\frac{1}{d-1}} \leq \prod_{i=1}^{d} f_i\left(\tilde{x}_i\right)^{\frac{1}{d-1}}.$$

Integrating with respect to x and applying Lemma 4.7.2 it follows that

$$\begin{aligned}\int_{\mathbb{R}^d}|u|^{\frac{d}{d-1}}(x)\,\mathrm{d}x &\leq \prod_{i=1}^{d}\left\|f_i^{\frac{1}{d-1}}\right\|_{L^{d-1}(\mathbb{R}^{d-1})}\\ &= \prod_{i=1}^{d}\|Du\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d-1}}\\ &= \|Du\|_{L^1(\mathbb{R}^d)}^{\frac{d}{d-1}}.\end{aligned}$$

Hence,

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} = \|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|Du\|_{L^1(\mathbb{R}^d)}.$$

Now using the density of $\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ it follows that

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq \|Du\|_{L^1(\mathbb{R}^d)} \tag{4.7.2}$$

for all $u \in \mathrm{W}^{1,1}\left(\mathbb{R}^d\right)$.
Step 2: Consider the case when $p \in (1, d)$.
Let $\gamma = \frac{p(d-1)}{d-p} > 1$. Then

$$\begin{aligned}
\left(\int_{\mathbb{R}^d} |u|^{p^*} \,\mathrm{d}x\right)^{\frac{1}{p^*}} &= \left(\int_{\mathbb{R}^d} |u|^{p^*} \,\mathrm{d}x\right)^{\frac{d-1}{d}} \left(\int_{\mathbb{R}^d} |u|^{p^*} \,\mathrm{d}x\right)^{-\frac{p-1}{p}} \\
&= \left(\int_{\mathbb{R}^d} |u|^{\frac{\gamma d}{d-1}} \,\mathrm{d}x\right)^{\frac{d-1}{d}} \left(\int_{\mathbb{R}^d} |u|^{\frac{\gamma d}{d-1}} \,\mathrm{d}x\right)^{-\frac{p-1}{p}} \\
&\overset{(4.7.2)}{\leq} \int_{\mathbb{R}^d} |D|u|^\gamma| \,\mathrm{d}x \left(\int_{\mathbb{R}^d} |u|^{\frac{\gamma d}{d-1}} \,\mathrm{d}x\right)^{-\frac{p-1}{p}} \\
&= \gamma \int_{\mathbb{R}^d} |u|^{\gamma-1}|Du| \,\mathrm{d}x \left(\int_{\mathbb{R}^d} |u|^{\frac{\gamma d}{d-1}} \,\mathrm{d}x\right)^{-\frac{p-1}{p}} \\
&\overset{\text{Hölder's}}{\leq} \gamma \left(\int_{\mathbb{R}^d} |u|^{(\gamma-1)\frac{p}{p-1}} \,\mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} |Du|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} |u|^{\frac{\gamma d}{d-1}} \,\mathrm{d}x\right)^{-\frac{p-1}{p}} \\
&= \gamma \left(\int_{\mathbb{R}^d} |Du|^p \,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

□

Remark 4.7.7.

1. *In words, Theorem 4.7.6 says that if a function is weakly differentiable and is in some* $\mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$*, for* $p < d$*, then it represents some function* $L^{p^*}\left(\mathbb{R}^d\right)$*, where* $p^* > p$*. However, Theorem 4.7.6 does not guarantee differentiability in* $L^{p^*}\left(\mathbb{R}^d\right)$*, it only guarantees that the function is* L^{p^*}*-integrable. Hence, Theorem 4.7.6 can be seen as trading differentiability for integrability.*

2. *Often one writes the embedding of Theorem 4.7.6 as* $\mathrm{W}^{1,p}\left(\mathbb{R}^d\right) \hookrightarrow L^{p^*}\left(\mathbb{R}^d\right)$*, to make explicit the fact that the embedding is continuous.*

Corollary 4.7.8 (Poincaré's Inequality)**.** *Let* $U \subseteq \mathbb{R}^d$ *be open and bounded. Let* $u \in \mathrm{W}_0^{1,p}(U)$ *for* $p \in [1, d)$*. Then,*

$$\|u\|_{L^q(U)} \leq c\|Du\|_{L^p(U)}$$

for all $q \in [1, p^*]$ *where* p^* *is the Sobolev conjugate of* p*.*

Proof. As $u \in \mathrm{W}_0^{1,p}(U)$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_c^\infty(U)$ such that $u_n \to u$ in $\mathrm{W}^{1,p}(U)$. In particular, $u_n \to u$ in $L^p(U)$, and so as U is bounded it follows by Hölder's inequality that $u_n \to u$ in $L^{p^*}(U)$ where $p^* = \frac{dp}{d-p} > p$. We can view $u_n \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ by taking it to be zero on $\mathbb{R}^d \setminus U$. Thus, applying the Gargliardo-Nirenberg-Sobolev inequality it follows that

$$\|u_n\|_{L^{p^*}(U)} = \|u_n\|_{L^{p^*}(\mathbb{R}^d)} \leq c\|Du_n\|_{L^p(\mathbb{R}^d)} = c\|Du_n\|_{L^p(U)},$$

where $c > 0$ is some constant independent of u. Passing to the limit it follows that

$$\|u\|_{L^{p^*}(U)} \leq c\|Du\|_{L^p(U)}. \tag{4.7.3}$$

For $q \in [1, p^*)$ let $\tilde{q} = \frac{q}{p^* - q} > 1$ so that $\frac{1}{q} = \frac{1}{p^*} + \frac{1}{\tilde{q}}$. Then, by applying Hölder's inequality it follows that

$$\|u\|_{L^q(U)} \leq \|u\|_{L^{p^*}(U)} \|\mathbf{1}\|_{L^{\tilde{q}}(U)} \overset{(4.7.3)}{\leq} \tilde{c} \|Du\|_{L^p(U)},$$

where we have used the fact that $\|\mathbf{1}\|_{L^{\tilde{q}}(U)} < \infty$ since U is bounded. We note that $\tilde{c}$ is independent of u, as for fixed q the value of $\|\mathbf{1}\|_{L^{\tilde{q}}(U)}$ is constant. □

Corollary 4.7.9. *Let $U \subseteq \mathbb{R}^d$ be open and bounded with a $\mathcal{C}^1$-boundary. Then for $p \in [1, d)$ and p^* its Sobolev conjugate, we have*

$$\mathrm{W}^{1,p}(U) \subseteq L^{p^*}(U).$$

In particular, there exists a constant c such that

$$\|u\|_{L^{p*}(U)} \leq c\|u\|_{\mathrm{W}^{1,p}(U)}$$

for all $u \in \mathrm{W}^{1,p}(U)$.

Proof. By the Theorem 4.5.2 there exists $\bar{u} := E(u) \in \mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$, such that $\bar{u}$ has compact support, $\bar{u} = u$ on U and

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^d)} \leq c\|u\|_{\mathrm{W}^{1,p}(U)}. \tag{4.7.4}$$

Since, $\bar{u}$ has compact support, using Theorem 4.4.8, there exists a sequence of functions $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ such that $u_n \to \bar{u}$ in $\mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$. Thus,

$$\|u_n\|_{L^{p^*}(\mathbb{R}^d)} \overset{\text{Thm } 4.7.6}{\leq} \tilde{c}\|Du_n\|_{L^p(\mathbb{R}^d)}.$$

In particular,

$$\|u_n - u_m\|_{L^{p^*}(\mathbb{R}^d)} \leq \tilde{c}\|Du_n - Du_m\|_{L^p(\mathbb{R}^d)}.$$

Which implies that $u_n \to \bar{u}$ in $L^{p^*}\left(\mathbb{R}^d\right)$ as well. Therefore,

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^d)} \leq \tilde{c}\|D\bar{u}\|_{L^p(\mathbb{R}^d)}. \tag{4.7.5}$$

Hence,

$$\|u\|_{L^{p^*}(U)} \leq \|\bar{u}\|_{L^{p^*}(U)} \overset{(4.7.5)}{\leq} \tilde{c}\|D\bar{u}\|_{L^p(\mathbb{R}^d)} \leq c\|\bar{u}\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \overset{(4.7.4)}{\leq} \bar{c}\|u\|_{\mathrm{W}^{1,p}(U)}.$$

□

Exercise 4.7.10. *Suppose there exists a function $u \in \mathcal{C}^2\left(\bar{U}\right)$ vanishing on ∂U, for which*

$$\frac{\int_U |\nabla u|^2(x)\,\mathrm{d}x}{\int_U u^2(x)\,\mathrm{d}x} \tag{4.7.6}$$

attains its minimum value λ. Then u is an eigenfunction with eigenvalue λ, namely

$$-\Delta u = \lambda u$$

in U. Moreover, λ is the smallest such eigenvalue with eigenfunction in $\mathcal{C}^2\left(\bar{U}\right)$.

Remark 4.7.11. *From Corollary 4.7.8 we know that*

$$\|u\|_{L^2(U)}^2 \leq C_p\|\nabla u\|_{L^2(U)}^2,$$

or equivalently

$$\frac{\int_U |\nabla u|^2 \,\mathrm{d}x}{\int_U |u|^2 \,\mathrm{d}x} \geq \frac{1}{C_p}.$$

Therefore, from Exercise 4.7.10 it follows that $C_p \geq \frac{1}{\lambda_1}$ where λ_1 is the smallest eigenvalue of $-\Delta$.

4.7.2 Morrey's Inequality

Theorem 4.7.6 deals with the case when $p \in [1, d)$. Now we will understand the case $p \in (d, \infty)$.

Exercise 4.7.12. *Show that $\mathcal{C}^{0,\frac{1}{2}}(\mathbb{R}) \subseteq \mathrm{H}^1(\mathbb{R})$. In particular, show that there exists a $c > 0$ such that*

$$\|u\|_{\mathcal{C}^{0,\frac{1}{2}}(\mathbb{R})} \leq c\|u\|_{\mathrm{H}^1(\mathbb{R})}.$$

Theorem 4.7.13 (Morrey's Inequality)**.** *On $\mathbb{R}^d$, let $p \in [d, \infty)$. Then there exists a constant $c = c(p,d)$ such that*

$$\|u\|_{\mathcal{C}^{0,\gamma}(\mathbb{R}^d)} \leq c\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}$$

for all $u \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, where $\gamma = 1 - \frac{d}{p}$.

Proof. Let Q be an open cube of side length r containing the origin. Then let

$$\bar{u} := \frac{1}{|Q|}\int_Q u(x)\,\mathrm{d}x.$$

Note that

$$\begin{aligned}
|\bar{u} - u(0)| &= \left|\frac{1}{|Q|}\int_Q u(x) - u(0)\,\mathrm{d}x\right| \\
&\leq \frac{1}{|Q|}\int_Q |u(x) - u(0)|\,\mathrm{d}x.
\end{aligned}$$

As $u \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, we can use the fundamental theorem of calculus to observe that

$$\begin{aligned}
u(x) - u(0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}u(tx)\,\mathrm{d}t \\
&= \sum_{i=1}^d \int_0^1 x_i \frac{\partial u}{\partial x_i}(tx)\,\mathrm{d}t.
\end{aligned}$$

Since $|x_i| < r$, for $x \in Q$, we have

$$|u(x) - u(0)| \leq r\int_0^1 \sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}(tx)\right|\,\mathrm{d}t.$$

Therefore,

$$\begin{aligned}
|\bar{u} - u(0)| &\leq \frac{r}{|Q|}\int_Q\int_0^1 \sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}(tx)\right|\,\mathrm{d}t\,\mathrm{d}x \\
&\overset{(1)}{\leq} \frac{r}{|Q|}\int_0^1 \left(\int_{tQ}\sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}(y)\right| t^{-d}\,\mathrm{d}y\right)\,\mathrm{d}t \\
&\overset{\text{Hölder's}}{\leq} \frac{r}{|Q|}\int_0^1 t^{-d}\left(\sum_{i=1}^d \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(tQ)} |tQ|^{\frac{1}{q}}\right)\,\mathrm{d}t
\end{aligned}$$

where in (1) we have changed the order of integration and set $y = tx$, and q is conjugate to p. Since $|tQ| = t^d r^d$, we can write

$$\begin{aligned}|\bar{u} - u(0)| &\leq c r^{1-d+\frac{d}{q}} \|Du\|_{L^p(\mathbb{R}^d)} \int_0^1 t^{-d+\frac{d}{q}} \,\mathrm{d}t \\ &= \frac{c}{1-\frac{d}{p}} r^{1-\frac{d}{p}} \|Du\|_{L^p(\mathbb{R}^d)} \\ &= c r^{\gamma} \|Du\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

where c is some constant. Take $x, y \in \mathbb{R}^d$ with $|x - y| = \frac{r}{2}$. Then pick a box of side length r containing x and y. By shifting the above result, and applying the triangle inequality, it follows that

$$|u(x) - u(y)| \leq |\bar{u} - u(x)| + |\bar{u} - u(y)| \leq c r^{\gamma} \|Du\|_{L^p(\mathbb{R}^d)}.$$

Hence,

$$\frac{|u(x) - u(y)|}{|x - y|^{\gamma}} \leq c 2^{\gamma} \|Du\|_{L^p(\mathbb{R}^d)}.$$

Taking the supremum over $x \neq y$ it follows that

$$[u]_{\mathcal{C}^{0,\gamma}(\mathbb{R}^d)} \leq c_{d,p} \|Du\|_{L^p(\mathbb{R}^d)}. \tag{4.7.7}$$

Next note that any $x \in \mathbb{R}^d$ belongs to some cube Q of side length 1, thus

$$|u(x)| \leq |\bar{u}| + |\bar{u} - u(x)| \leq |\bar{u}| + c \|Du\|_{L^p(\mathbb{R}^d)}.$$

As

$$|\bar{u}| \leq \int_Q |u(x)| \,\mathrm{d}x \overset{\text{Hölder's}}{\leq} \|u\|_{L^p(\mathbb{R}^d)} \|\mathbf{1}\|_{L^q(Q)}$$

we have that

$$|u(x)| \leq \tilde{c} \left(\|u\|_{L^p(\mathbb{R}^d)} + \|Du\|_{L^p(\mathbb{R}^d)} \right)$$

for some constant $\tilde{c}$ independent of x. Hence,

$$\|u\|_{\mathcal{C}^0(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} |u(x)| \leq \tilde{c} \|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}. \tag{4.7.8}$$

From (4.7.7) and (4.7.8) it follows that

$$\|u\|_{\mathcal{C}^{0,\gamma}(\mathbb{R}^d)} \leq \bar{c} \|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}.$$

$\square$

Remark 4.7.14. *The fundamental theorem of calculus is critical for Sobolev embeddings as it relates a function to the integral of its derivative, thus trading differentiability for integrability.*

Corollary 4.7.15. *Suppose $u \in \mathrm{W}^{1,p}(U)$ for $U \subseteq \mathbb{R}^d$ open and bounded with $\mathcal{C}^1$-boundary. Let $p \in (d, \infty)$ and $\gamma = 1 - \frac{d}{p}$. Then there exists a $u^* \in \mathcal{C}^{0,\gamma}(U)$ such that $u = u^*$ almost everywhere and*

$$\|u^*\|_{C^{0,\gamma}(U)} \leq c \|u\|_{\mathrm{W}^{1,p}(U)}$$

for some constant c independent of u. That is, there exists a continuous embedding $\mathrm{W}^{1,p}(U) \hookrightarrow \mathcal{C}^{0,\gamma}(U)$.

Proof. As ∂U is $\mathcal{C}^1$, using Theorem 4.5.2 there exists an extension $Eu = \bar{u} \in \mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$ such that $\bar{u}|_U = u$, $\bar{u}$ has compact support and

$$\|\bar{u}\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \leq c \|u\|_{\mathrm{W}^{1,p}(U)} \tag{4.7.9}$$

Then since $\bar{u}$ has compact support, it follows by Theorem 4.4.3 there exists a sequence of functions $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ such that

$$u_n \to \bar{u} \tag{4.7.10}$$

in $\mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$. From Theorem 4.7.13 we have that

$$\|u_n\|_{\mathcal{C}^{0,1-\frac{d}{p}}(\mathbb{R}^d)} \leq c\|u_n\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}, \tag{4.7.11}$$

for all $n \geq 1$. In particular,

$$\|u_n - u_m\|_{\mathcal{C}^{0,1-\frac{d}{p}}(\mathbb{R}^d)} \leq c\|u_n - u_m\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}$$

for all $n, m \geq 1$. Hence, by the completeness of $\mathcal{C}^{0,1-\frac{d}{p}}\left(\mathbb{R}^d\right)$ there exists a $u^* \in \mathcal{C}^{0,1-\frac{d}{p}}\left(\mathbb{R}^d\right)$ such that

$$u_n \to u^* \tag{4.7.12}$$

in $\mathcal{C}^{0,1-\frac{d}{p}}\left(\mathbb{R}^d\right)$. From (4.7.10) and (4.7.12) it follows that $u^* = u$ almost everywhere on U. Moreover, from (4.7.11) we have

$$\|u^*\|_{\mathcal{C}^{0,1-\frac{d}{p}}(\mathbb{R}^d)} \leq c\|\bar{u}\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}.$$

Therefore using (4.7.9) we conclude that,

$$\|u^*\|_{\mathcal{C}^{0,1-\frac{d}{p}}(\mathbb{R}^d)} \leq \tilde{c}\|u\|_{\mathrm{W}^{1,p}(U)}.$$

□

For $U \subseteq \mathbb{R}^d$ open and bounded with a $\mathcal{C}^1$-boundary we have thus far shown the following statements.

1. If $p \in [1, d)$ then $\mathrm{W}^{1,p}(U) \hookrightarrow L^{p^*}(U)$ is a continuous embedding, where $\frac{1}{p^*} + \frac{1}{d} = \frac{1}{p}$.
2. If $p \in (d, \infty)$ then $\mathrm{W}^{1,p}(U) \hookrightarrow \mathcal{C}^{0,\gamma}(U)$ is a continuous embedding, where $\gamma = 1 - \frac{d}{p} < 1$.

Applying these embeddings iteratively establishes similar embeddings for higher-order Sobolev spaces. As $p^* > p$, the embedding $\mathrm{W}^{1,p}(U) \hookrightarrow L^{p^*}(U)$ trades integrability for differentiability. However, if there is some differentiability remaining we can continue to trade it away. In particular, if a function is in $\mathrm{W}^{k,p}$, for k sufficiently large, then eventually $p^* > d$, at which point we can use the embedding $\mathrm{W}^{1,p}(U) \hookrightarrow \mathcal{C}^{0,\gamma}(U)$ to arrive at a Hölder continuous, and thus continuous, function. Applying the same procedure for the derivatives, it may be that the resulting embedding is also $\mathcal{C}^2$, provided k is sufficiently large. Hence, we can arrive at classical solutions to partial differential equations. Thus, if we can show the integrability of a solution to a partial differential equation, we can then use Sobolev embeddings to show that they are regular and classical solutions to the partial differential equation.

4.7.3 General Sobolev Inequality

We can generalise and consolidate the previous inequalities with Theorem 4.7.16

Theorem 4.7.16. *Let $U \subseteq \mathbb{R}^d$ be an open and bounded set with $\mathcal{C}^1$ boundary. Let $u \in \mathrm{W}^{k,p}(U)$.*

1. *If $k < \frac{d}{p}$ then $u \in L^q(U)$ where $\frac{1}{q} = \frac{1}{p} - \frac{k}{d}$. Moreover,*

$$\|u\|_{L^q(U)} \leq C\|u\|_{\mathrm{W}^{k,p}(U)}$$

   where $C = C(k, p, d, U)$.

2. *If $k > \frac{d}{p}$ then $u \in \mathcal{C}^{k-\left[\frac{d}{p}\right]-1,\gamma}$, where $[\cdot]$ denotes the integer part and*

$$\gamma = \begin{cases} \left[\frac{d}{p}\right] + 1 - \frac{d}{p} & \frac{d}{p} \notin \mathbb{Z} \\ \text{any positive number less than } 1 & \frac{d}{p} \in \mathbb{Z}. \end{cases}$$

Moreover,

$$\|u\|_{\mathcal{C}^{k-\left[\frac{d}{p}\right]-1,\gamma}(U)} \leq C\|u\|_{\mathrm{W}^{k,p}(U)},$$

for $C = C(k,p,d,\gamma,U)$.

Proof.

1. Since, $D^\alpha u \in L^p(U)$ for $|\alpha| \leq k-1$, it follows by Theorem 4.7.6 that

$$\|D^\alpha u\|_{L^{p^*}(U)} \leq C\|u\|_{\mathrm{W}^{k,p}(U)}$$

which implies that $u \in \mathrm{W}^{k-1,p^*}(U)$. Similarly, $u \in \mathrm{W}^{k-2,p^{**}}(U)$ where $\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{d} = \frac{1}{p} - \frac{2}{d}$. Eventually, we deduce that $u \in \mathrm{W}^{0,q}(U) = L^q(U)$ where $\frac{1}{q} = \frac{1}{p} - \frac{k}{d}$ with an estimate

$$\|u\|_{L^q(U)} \leq C\|u\|_{\mathrm{W}^{k,p}(U)}$$

holding for $C = C(k,p,d,U)$.

2. - Suppose $\frac{d}{p} \notin \mathbb{Z}$. Then as above, $u \in \mathrm{W}^{k-l,r}(U)$ where $\frac{1}{r} = \frac{1}{p} - \frac{l}{d}$ provided $lp < d$. In particular, let $l = \left[\frac{d}{p}\right]$, so that $l < \frac{d}{p} < l+1$, then $r = \frac{pd}{d-pl} > d$. Therefore, as $u \in \mathrm{W}^{k-l,r}(U)$ and using Theorem 4.7.13, we deduce that $D^\alpha u \in \mathcal{C}^{0,1-\frac{d}{r}}\left(\bar{U}\right)$ for every $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leq k-l-1$. Moreover,

$$1 - \frac{d}{r} = 1 - \frac{d}{p} + l = \left[\frac{d}{p}\right] + 1 - \frac{d}{p}.$$

Thus, $u \in \mathcal{C}^{k-\left[\frac{d}{p}\right]-1,\left[\frac{d}{p}\right]+1-\frac{d}{p}}\left(\bar{U}\right)$ with the required estimate.

   - Suppose $\frac{d}{p} \in \mathbb{Z}$. Let $l = \frac{d}{p} - 1$. Then as above, $u \in \mathrm{W}^{k-l,r}(U)$ for $r = \frac{pd}{d-pl} = d$. Using Theorem 4.7.6 we have that $D^\alpha u \in L^q(U)$ for $q \in [d,\infty)$ and all $|\alpha| \leq k-l-1 = k - \frac{d}{p}$. Therefore, from Theorem 4.7.13 we have $D^\alpha u \in \mathcal{C}^{0,1-\frac{d}{q}}\left(\bar{U}\right)$ for all $q \in (d,\infty)$ and $|\alpha| \leq k - \frac{d}{p} - 1$. Therefore, $u \in \mathcal{C}^{k-\frac{d}{p}-1,\gamma}\left(\bar{U}\right)$ for $\gamma \in (0,1)$ with the required estimate.

□

Moreover, Proposition 4.7.17 deals with the case when $d = p$.

Proposition 4.7.17. *For $d = p$, we have*

$$\mathrm{W}^{1,p}\left(\mathbb{R}^d\right) \subseteq L^q\left(\mathbb{R}^d\right)$$

for all $q \in [p,\infty)$. In particular, for fixed q there exists a $c_q > 0$ such that

$$\|u\|_{L^q(\mathbb{R}^d)} \leq c_q\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}.$$

Remark 4.7.18. *In Proposition 4.7.17 it is important that $q \neq \infty$. Indeed, consider*

$$u(x) = \log\left(\log\left(1 + \frac{1}{|x|}\right)\right).$$

Then $u \in \mathrm{W}^{1,d}(U)$, where $U = B_1(0)$, however, $u \notin L^\infty(U)$.

4.8 Solution to Exercises

Exercise 4.1.10

Solution. Let $\lambda \in \mathbb{R}$, then

$$\begin{aligned}
[\lambda u]_{\mathcal{C}^{0,\gamma}(\bar{U})} &= \sup_{x,y\in U} \frac{|(\lambda u)(x) - (\lambda u)(y)|}{|x-y|^\gamma} \\
&= |\lambda| \sup_{x,y\in U} \frac{|u(x) - u(y)|}{|x-y|^\gamma} \\
&= |\lambda| [u]_{\mathcal{C}^{0,\gamma}(\bar{U})}.
\end{aligned}$$

Moreover, for $u, v \in \mathcal{C}^{0,\gamma}\left(\bar{U}\right)$ we have

$$\begin{aligned}
[u+v]_{\mathcal{C}^{0,\gamma}(\bar{U})} &= \sup_{x,y\in U} \frac{|(u+v)(x) - (u+v)(y)|}{|x-y|^\gamma} \\
&\le \sup_{x,y\in U} \frac{|u(x) - (y)| + |v(x) - v(y)|}{|x-y|^\gamma} \\
&= [u]_{\mathcal{C}^{0,\gamma}(\bar{U})} + [v]_{\mathcal{C}^{0,\gamma}(\bar{U})}.
\end{aligned}$$

Therefore, $[\cdot]_{\mathcal{C}^{0,\gamma}(\bar{U})}$ is a semi-norm on $\mathcal{C}^{0,\gamma}\left(\bar{U}\right)$. It is not a norm as for $u \equiv \mathbf{c} \in \mathbb{R}^d \setminus \{0\}$ we have $[u]_{\mathcal{C}^{0,\gamma}(\bar{U})} = 0$. □

Exercise 4.4.10

Solution. ($\Rightarrow$). If $u \in \mathcal{C}_c^k\left(\mathbb{R}^d\right)$. Then, $\widehat{D^\alpha u} = (iy)^\alpha \hat{u}$. Thus, by approximating $u \in \mathrm{H}^k\left(\mathbb{R}^d\right)$ by compactly supported functions it follows that $\widehat{D^\alpha u} = (iy)^\alpha \hat{u}$ for $u \in \mathrm{H}^k\left(\mathbb{R}^d\right)$. In particular,

$$\int_{\mathbb{R}^d} |y|^{2k} |\hat{u}|^2 \,\mathrm{d}y = 2\pi \int_{\mathbb{R}^d} \left| D^{(k,0,\dots,0)} u \right|^2 \,\mathrm{d}x \le c \int_{\mathbb{R}^d} \left| D^k u \right|^2 \,\mathrm{d}x.$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(1 + |y|^k\right)^2 |\hat{u}|^2 \,\mathrm{d}y &\le c \int_{\mathbb{R}^d} \left(1 + |y|^{2k}\right) |\hat{u}|^2 \,\mathrm{d}y \\
&\le c \left(\|u\|_{L^2(\mathbb{R}^d)} + \left\| D^k u \right\|_{L^2(\mathbb{R}^d)} \right) \\
&= c \|u\|_{\mathrm{H}^k(\mathbb{R}^d)} \\
&< \infty.
\end{aligned}$$

Therefore, $\left(1 + |y|^k\right) \hat{u} \in L^2\left(\mathbb{R}^d\right)$.
($\Leftarrow$). Note that

$$\|(iy)^\alpha \hat{u}\|_{L^2(\mathbb{R}^d)} \le c \left\| \left(1 + |y|^k\right)^2 \hat{u} \right\|_{L^2(\mathbb{R}^d)}. \tag{4.8.1}$$

Let $u_\alpha = \mathcal{F}^{-1}\left((iy)^\alpha \hat{u}\right)$. For $\varphi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \bar{u}_\alpha \varphi \,\mathrm{d}x &= \int_{\mathbb{R}^d} \hat{\bar{u}}_\alpha \hat{\varphi} \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \bar{\hat{u}}_\alpha \hat{\varphi} \,\mathrm{d}x \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^d} (iy)^\alpha \bar{\hat{u}} \hat{\varphi} \,\mathrm{d}x \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^d} \left(\widehat{D^\alpha \varphi}\right) \bar{\hat{u}} \,\mathrm{d}x \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^d} \left(D^\alpha \varphi\right) \bar{u} \,\mathrm{d}x.
\end{aligned}$$

Where the bar notation is complex conjugation. It follows that $u_\alpha = D^\alpha u$ in the weak sense. Thus since

$$\|u_\alpha\|_{L^2(\mathbb{R}^d)} = 2\pi \left\|(iy)^\alpha \hat{u}\right\|_{L^2(\mathbb{R}^d)},$$

it follows by (4.8.1) that $D^\alpha u \in L^2\left(\mathbb{R}^d\right)$. Therefore, $u \in \mathrm{H}^k(U)$. ☐

Exercise 4.5.1

Solution. Using the change of variables formula it follows that

$$\begin{aligned}\|u \circ \Phi\|^p_{L^p(\Psi(U))} &= \int_U |u|^p |D\Psi| \,\mathrm{d}x \\ &\leq \sup_U |D\Psi| \|u\|^p_{L^p(U)}.\end{aligned} \tag{4.8.2}$$

Through the chain rule we have

$$D(u \circ \Phi)(x) = Du(\Phi(x))D\Phi(x),$$

thus

$$|D(u \circ \Phi)(x)| \leq |Du(\Phi(x))|\|D\Phi\| \leq c|Du(\Phi(x))|.$$

Therefore,

$$\begin{aligned}\|D(u \circ \Phi)\|^p_{L^p(\Psi(U))} &\leq c \int_{\Psi(U)} |D(u \circ \phi)|^p \,\mathrm{d}x \\ &= c \int_U |Du|^p |D\Psi| \,\mathrm{d}x \\ &\leq c\|Du\|^p_{L^p(U)} \sup_U |D\Psi|\end{aligned} \tag{4.8.3}$$

Combining (4.8.2) and (4.8.3) it follows that

$$\|u \circ \Phi\|_{\mathrm{W}^{1,p}(\Psi(U))} \leq c\|u\|_{\mathrm{W}^{1,p}(U)}$$

for some constant dependent only on Φ and Ψ. Note that we have implicitly used the fact that U is bounded, Φ,Ψ are $\mathcal{C}^1$ and Φ, Ψ, $D\Phi$ and $D\Psi$ are bounded on U. Moreover, we have used the fact that $u \circ \Phi \in \mathcal{C}^1\left(\bar{U}\right) \subseteq \mathrm{W}^{1,p}(U)$. ☐

Exercise 4.7.10

Proof. Fix $\phi \in \mathcal{C}_c^\infty(U)$. Consider $\Psi : (-\epsilon, \epsilon) \to \mathbb{R}$ given by

$$\Psi(t) = \frac{\int_U |\nabla(u + t\phi)|^2 \,\mathrm{d}x}{\int_U (u + t\phi)^2 \,\mathrm{d}x}.$$

For sufficiently small $\epsilon > 0$ the function $u + t\phi$ is non-zero and so Ψ is well-defined. Moreover, Ψ is differentiable. In particular, the minimum of Ψ is achieved at $t = 0$ with $\Psi(0) = \lambda$. Therefore,

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int_U |\nabla(u + t\phi)|^2 \,\mathrm{d}x}{\int_U (u + t\phi)^2 \,\mathrm{d}x}\right)\Bigg|_{t=0} \\ &\overset{(1)}{=} 2\frac{\int_U \langle \nabla u, \nabla \phi \rangle \,\mathrm{d}x}{\int_U u^2 \,\mathrm{d}x} - 2\frac{\int_U |\nabla u|^2 \,\mathrm{d}x \int_U u\phi \,\mathrm{d}x}{\left(\int_U u^2 \,\mathrm{d}x\right)^2} \\ &= 2\frac{\int_U \langle \nabla u, \nabla \phi \rangle \,\mathrm{d}x}{\int_U u^2 \,\mathrm{d}x} - 2\frac{\int_U \lambda u\phi \,\mathrm{d}x}{\int_U u^2 \,\mathrm{d}x} \\ &\overset{(2)}{=} -2\frac{\int_U \Delta u \, \phi \,\mathrm{d}x}{\int_U u^2 \,\mathrm{d}x} - 2\frac{\int_U \lambda u\phi \,\mathrm{d}x}{\int_U u^2 \,\mathrm{d}x} \\ &= -2\frac{\int_U (\Delta u + \lambda u)\phi \,\mathrm{d}x}{\int_U u^2 \,\mathrm{d}x}\end{aligned}$$

where in (1) we have used the quotient rule, and in (2) we have integrated by parts. Since $\phi \in \mathcal{C}_c^\infty(U)$ was arbitrary, it follows that

$$\Delta u + \lambda u = 0$$

as required. Suppose that there exists a $\mu < \lambda$ such that

$$\Delta u + \mu u = 0 \tag{4.8.4}$$

and $u \in \mathcal{C}^2\left(\bar{U}\right)$ is non-zero and vanishing on ∂U. Then, multiplying (4.8.4) by u and integrating by parts it follows that

$$\int_U |\nabla u|^2 \, \mathrm{d}x = \mu \int_U u^2(x) \, \mathrm{d}x,$$

which contradicts that the quotient (4.7.6) has minimum value λ. □

Exercise 4.7.12

Solution. Let $u \in \mathcal{C}_c^\infty(\mathbb{R})$. Then $(u|u|)' = 2|u|u'$ and so

$$|u(x)|u(x) = 2\int_{-\infty}^{x} |u(y)|u'(y) \, \mathrm{d}y$$

which implies that

$$u(x)^2 \lesssim \|u\|_{L^2(\mathbb{R})} \|u'\|_{L^2(\mathbb{R})}.$$

Hence, using Lemma 4.6.1 it follows that

$$\|u\|_{L^\infty(\mathbb{R})} \lesssim \left(\|u\|_{L^2(\mathbb{R})}^2 + \|u'\|_{L^2(\mathbb{R})}^2\right)^{\frac{1}{2}} = \|u\|_{\mathrm{H}^1(\mathbb{R})}.$$

Moreover, as

$$u(x) - u(y) = \int_x^y u'(t) \, \mathrm{d}t$$

we get that

$$|u(x) - u(y)| \lesssim |x-y|^{\frac{1}{2}} \|u'\|_{L^2(\mathbb{R})}.$$

Hence,

$$\frac{|u(x) - u(y)|}{\sqrt{|x-y|}} \lesssim \|u'\|_{L^2(\mathbb{R})},$$

which upon taking supremums means that

$$[u]_{\mathcal{C}^{0,\frac{1}{2}}(\mathbb{R})} = \sup_{x,y\in\mathbb{R}, x\neq y} \frac{|u(x) - u(y)|}{\sqrt{|x-y|}} \lesssim \|u'\|_{L^2(\mathbb{R})} \lesssim \|u\|_{\mathrm{H}^1(\mathbb{R})}.$$

Thus, it follows that

$$\|u\|_{\mathcal{C}^{0,\frac{1}{2}}(\mathbb{R})} \lesssim \|u\|_{\mathrm{H}^1(\mathbb{R})}.$$

Using the density of $\mathcal{C}_c^\infty(\mathbb{R})$, the result holds for $u \in \mathrm{H}^1(\mathbb{R})$. □

5 Second-Order Elliptic Boundary Value Problems

Throughout, $U \subseteq \mathbb{R}^d$ will be open with $\mathcal{C}^1$ boundary.

5.1 Elliptic Operators

With $u \in \mathcal{C}^2\left(\bar{U}\right)$ let

$$Lu := -\sum_{i,j=1}^{d} \left(a_{ij}u_{x_i}\right)_{x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu, \tag{5.1.1}$$

where a_{ij}, b, c are given. Without loss of generality, we can assume that $a_{ij} = a_{ji}$ for all i, j.

Remark 5.1.1.

1. *A partial differential equation in the form of* (5.1.1) *is said to be in divergence form as the highest order term is of the form* $\nabla \cdot (A\nabla u)$.
2. *If* $a_{ij} \in \mathcal{C}^1(U)$, *then* (5.1.1) *can be written as*

$$Lu = -\sum_{i,j=1}^{d} a_{ij}u_{x_i x_j} + \sum_{i=1}^{d} \tilde{b}_i u_{x_i} + cu. \tag{5.1.2}$$

   In such a case, the partial differential equation in non-divergence form.

Definition 5.1.2. *Let* L *be a partial differential operator.*

1. L *is elliptic if*

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j > 0$$

   for every $x \in U$ *and* $\xi \in \mathbb{R}^d \setminus \{0\}$.

2. L *is uniformly elliptic if*

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \theta\|\xi\|^2$$

   for every $x \in U$, $\xi \in \mathbb{R}^d \setminus \{0\}$ *and* $\theta > 0$ *independent of* x.

Remark 5.1.3. *Ellipticity means that the matrix* $A(x) = (a_{ij}(x))$ *is positive definite, and uniformly elliptic if the smallest eigenvalue is strictly greater than zero. Indeed, statement 2 of Definition 5.1.2 can be re-written as*

$$Q(\xi) := \xi^\top A\xi \geq \theta\|\xi\|^2.$$

5.2 The Weak Formulation

Consider

$$\begin{cases} Lu = f & U \\ u|_{\partial U} = 0 & \partial U. \end{cases} \tag{5.2.1}$$

Let $u \in \mathcal{C}^2\left(\bar{U}\right)$, and assume u solves (5.2.1). Take any $v \in \mathcal{C}^2\left(\bar{U}\right)$ with $v|_{\partial U} = 0$, then

$$\begin{aligned}
\int_U fv \,\mathrm{d}x &= \int_U Lu \cdot v \,\mathrm{d}x \\
&= \int_U v \left(-\sum_{i,j=1}^{d} (a_{ij}u_{x_i})_{x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu \right) \mathrm{d}x \\
&= \int_{\partial U} v \left(-\sum_{i,j=1}^{d} a_{ij}u_{x_i} \right) + \int_U \sum_{i,j=1}^{d} a_{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{d} b_i u_{x_i} v + cuv \,\mathrm{d}x \\
&= \int_U \sum_{i,j=1}^{d} a_{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{d} b_i u_{x_i} v + cuv \,\mathrm{d}x.
\end{aligned}$$

Hence,

$$\int_U fv \,\mathrm{d}x = \int_U \left(\sum_{i,j=1}^{d} a_{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{d} b_i u_{x_i} v + cuv \right) \mathrm{d}x, \tag{5.2.2}$$

for all $v \in \mathcal{C}^2\left(\bar{U}\right)$ with $v|_{\partial U} = 0$. Let

$$B[u,v] := \int_U fv \,\mathrm{d}x = \int_U \left(\sum_{i,j=1}^{d} a_{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{d} b_i u_{x_i} v + cuv \right) \mathrm{d}x. \tag{5.2.3}$$

Conversely, if $u \in \mathcal{C}^2\left(\bar{U}\right)$ with $u|_{\partial U} = 0$ is such that (5.2.2) holds for all $v \in \mathcal{C}^2\left(\bar{U}\right)$ with $v|_{\partial U} = 0$, then

$$\int_U (f - Lu)v \,\mathrm{d}x = 0$$

for all $v \in \mathcal{C}^2\left(\bar{U}\right)$ with $v|_{\partial U} = 0$. Thus, $Lu = f$ and so u solves (5.2.1).

Remark 5.2.1. *Equation* (5.2.2) *is referred to as the weak formulation of* (5.2.1)*. In particular, we have shown that $u \in \mathcal{C}^2\left(\bar{U}\right)$ satisfies* (5.2.1) *if and only if u satisfies the weak formulation* (5.2.1)*. This is useful, as* (5.2.2) *makes sense for $u \in \mathrm{H}_0^1(U) = \mathrm{W}^{1,2}(U)$.*

Definition 5.2.2. *A function $u \in \mathrm{H}_0^1(U)$ is a weak solution to* (5.2.1) *if $f \in L^2(U)$, and*

$$B[u,v] = (f,v)_{L^2(U)}$$

for all $v \in \mathrm{H}_0^1(U)$.

5.3 Existence of Weak Solutions

5.3.1 Lax-Milgram

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. With $\langle\cdot,\cdot\rangle$ we denote the pairing of H with its dual space. Where we recall that the dual space of H, denoted H^*, is the collection of all bounded linear functionals on H.

Theorem 5.3.1 (Riesz Representation)**.** *Let H be a Hilbert space. Then for all $f \in H^*$ there exists a unique $\varphi \in H$ such that*

$$\langle f, x\rangle := f(x) = (\varphi, x)$$

for all $x \in H$.

Theorem 5.3.2 (Lax-Milgram)**.** *Let $B : H \times H \to \mathbb{R}$ be a bilinear map for which there exists a constant $\alpha > 0$ such that*

$$|B[u, v]| \leq \alpha \|u\| \|v\| \tag{5.3.1}$$

for every $u, v \in H$. Moreover, there exists a constant $\beta > 0$ such that

$$|B[u, u]| \geq \beta \|u\|^2 \tag{5.3.2}$$

for all $u \in H$. Then if $f \in H^*$, there exists a unique $u \in H$ such that

$$B[u, v] = \langle f, v \rangle$$

for all $v \in H$.

Remark 5.3.3.

1. *Equation* (5.3.1) *says that B is bounded, and* (5.3.2) *says it is coercive.*
2. *Since B is a pairing of H with itself, we can consider the map $\Phi : H \to H^*$ given by $v \mapsto B[v, \cdot]$. By Theorem 5.3.1, for all $f \in H^*$ we can write $f(\cdot) = (w, \cdot)$ for some $w \in H$. Thus if Φ is bijective there exists a $v \in H$ such that $B[v, \cdot] = (w, \cdot) = f(\cdot)$ for some $w \in H$. Hence, w is a weak solution. This means that to determine the existence of solutions it suffices to understand when the map is bijective.*
3. *Theorem 5.3.2 is a generalisation of Theorem 5.3.1 as $B[\cdot, \cdot]$ is not necessarily an inner product, indeed it does not have to be symmetric. If B were symmetric, then*

   $$((u, v)) := B[u, v]$$

   is an inner product on H. Applying Theorem 5.3.1 to $(H, B[\cdot, \cdot])$ yields Theorem 5.3.2.

5.3.2 Energy Estimate

To leverage Theorem 5.3.2 to determine when weak solutions (5.2.2), it is necessary to verify its assumptions for $B[\cdot, \cdot]$.

Theorem 5.3.4. *Assume $a_{ij} = a_{ji}, b_i, c \in L^\infty(U)$ and L is uniformly elliptic. Then for B as given by* (5.2.3)*, there exists $\alpha, \beta > 0$ and $\gamma \geq 0$ such that*

$$|B[u, v]| \leq \alpha \|u\|_{\mathrm{H}_0^1(U)} \|v\|_{\mathrm{H}_0^1(U)}$$

and

$$\beta \|u\|^2_{\mathrm{H}_0^1(U)} \leq B[u, u] + \gamma \|u\|^2_{L^2(U)} \tag{5.3.3}$$

for all $u, v \in \mathrm{H}_0^1(U)$.

Proof. Observe that,

$$\begin{aligned}
|B[u, v]| &\leq \sum_{i,j=1}^{d} \|a_{ij}\|_{L^\infty(U)} \int_U |Du||Dv| \,\mathrm{d}x + \sum_{j=1}^{d} \|b_i\|_{L^\infty(U)} \int_U |Du||v| \,\mathrm{d}x + \|c\|_{L^\infty(U)} \int_U |uv| \,\mathrm{d}x \\
&\overset{\text{Hölder's}}{\leq} c_1 \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)} + c_2 \|Du\|_{L^2(U)} \|v\|_{L^2(U)} + c_3 \|u\|_{L^2(U)} \|v\|_{L^2(U)} \\
&\leq \tilde{c} \left(\|u\|_{L^2(U)} + \|Du\|_{L^2(U)} \right) \left(\|v\|_{L^2(U)} + \|Dv\|_{L^2(U)} \right) \\
&\leq \alpha \|u\|_{\mathrm{H}_0^1(U)} \|v\|_{\mathrm{H}^1(U)}.
\end{aligned}$$

Using the ellipticity assumption, we have

$$\begin{aligned}
\theta \int_U |Du|^2 \,\mathrm{d}x &\le \int_U \sum_{i,j}^d a_{ij} u_{x_i} u_{x_j} \,\mathrm{d}x \\
&= B[u,u] - \int_U \sum_{i=1}^d b_i u_{x_i} u + cu^2 \,\mathrm{d}x \\
&\le B[u,u] + \sum_{i=1}^d \|b_i\|_{L^\infty(U)} \int_U |Du||u| \,\mathrm{d}x + \|c\|_{L^\infty(U)} \int_U u^2 \,\mathrm{d}x \\
&\le B[u,u] + \sum_{i=1}^d \|b_i\|_{L^\infty(U)} \left(\epsilon \int_U |Du|^2 \,\mathrm{d}x + \frac{1}{4\epsilon} \int_U u^2 \,\mathrm{d}x \right) \\
&\quad + \|c\|_{L^\infty(U)} \int_U u^2 \,\mathrm{d}x,
\end{aligned}$$

for $\epsilon > 0$, where the last inequality is an application of $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ for $a, b \in \mathbb{R}$. Choosing ϵ such that

$$\epsilon \sum_{i=1}^d \|b_i\|_{L^\infty(U)} < \frac{\theta}{2}$$

it follows that

$$\frac{\theta}{2} \int_U |Du|^2 \,\mathrm{d}x \le B[u,u] + \tilde{c} \int_U u^2 \,\mathrm{d}x$$

for some $\tilde{c} > 0$. Therefore,

$$\frac{\theta}{2} \|u\|^2_{\mathrm{H}^1_0(U)} = \frac{\theta}{2} \left(\|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)} \right) \le B[u,u] + \gamma \|u\|^2_{L^2(U)}$$

as required. □

Remark 5.3.5.

1. *Inequality* (5.3.3) *is known as Gårding's inequality.*
2. *Note that when* $b_i = 0$ *for all* $i \in \{1, \dots, d\}$ *and* $c \ge 0$, *we have*

$$\theta \|Du\|^2_{L^2(U)} \le B[u,u].$$

So that by Corollary 4.7.8 we get (5.3.3) *with* $\gamma = 0$. *Consequently, we can apply Theorem 5.3.2 to solve* (5.2.1), *which in this case reduces to Laplace's equation.*

Theorem 5.3.6. *Let* $U \subseteq \mathbb{R}^d$ *be open with* $\mathcal{C}^1$*-boundary and let* L *be as given by* (5.1.2). *Then there exists a* $\gamma \ge 0$ *such that for any* $\mu \ge \gamma$ *and* $f \in L^2(U)$ *there exists a unique weak solution* $u \in \mathrm{H}^1_0(U)$ *to*

$$\begin{cases} Lu + \mu u = f & U \\ u = 0 & \partial U. \end{cases} \tag{5.3.4}$$

Moreover,

$$\|u\|_{\mathrm{H}^1(U)} \le c \|f\|_{L^2(U)}$$

for some $c = c(L, U) \ge 0$.

Proof. Using Theorem 5.3.4 there exists $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

$$|B[u,v]| \leq \alpha \|u\|_{\mathrm{H}^1(U)} \|v\|_{\mathrm{H}^1(U)} \tag{5.3.5}$$

and

$$\beta \|u\|^2_{\mathrm{H}^1(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}. \tag{5.3.6}$$

For $\mu \geq \gamma$ let $L_\mu = L + \mu$ and consider the bilinear form

$$B_\mu[u,v] := B[u,v] + \mu(u,v)_{L^2(U)}.$$

Then,

$$\begin{aligned}
|B_\mu[u,v]| &\leq |B[u,v]| + \left|\mu(u,v)_{L^2(U)}\right| \\
&\overset{(5.3.5)}{\leq} \alpha \|u\|_{\mathrm{H}^1(U)} \|v\|_{\mathrm{H}^1(U)} + \mu(u,v)_{L^2(U)} \\
&\overset{\text{Hölder's}}{\leq} \alpha \|u\|_{\mathrm{H}^1(U)} \|v\|_{\mathrm{H}^1(U)} + \mu \|u\|_{L^2(U)} \|v\|_{L^2(U)} \\
&\leq \alpha \|u\|_{\mathrm{H}^1(U)} \|v\|_{\mathrm{H}^1(U)} + \mu \|u\|_{\mathrm{H}^1(U)} \|v\|_{\mathrm{H}^1(U)} \\
&= \tilde{\alpha} \|u\|_{\mathrm{H}^1(U)} \|v\|_{\mathrm{H}^1(U)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\beta \|u\|^2_{\mathrm{H}^1(U)} &\overset{(5.3.6)}{\leq} B[u,u] + \gamma \|u\|^2_{L^2(U)} \\
&\overset{\mu \geq \gamma}{\leq} B_\mu[u,u].
\end{aligned} \tag{5.3.7}$$

Therefore, $B_\mu[\cdot,\cdot]$ satisfies the conditions of Theorem 5.3.2. In particular, fix $f \in L^2(U)$ and set $\langle f, v \rangle := (f,v)_{L^2(U)}$. This is a bounded linear functional on $L^2(U)$, and thus a bounded and linear functional on $\mathrm{H}^1_0(U)$. Applying Theorem 5.3.2, there exists a unique $u \in \mathrm{H}^1_0(U)$ such that

$$B_\mu[u,\cdot] = \langle f, \cdot \rangle$$

on $\mathrm{H}^1_0(U)$. That is, u is the unique solution to (5.3.4). Moreover, using

$$\begin{aligned}
\beta \|u\|^2_{\mathrm{H}^1(U)} &\overset{(5.3.7)}{\leq} B_\mu[u,u] \\
&= (f,u)_{L^2(U)} \\
&\overset{\text{Hölder's}}{\leq} \|f\|_{L^2(U)} \|u\|_{L^2(U)},
\end{aligned}$$

therefore,

$$\beta \|u\|_{\mathrm{H}^1(U)} \leq \|f\|_{L^2(U)}.$$

$\square$

Remark 5.3.7.

1. *Theorem 5.3.6 provides a solution to a boundary value problem, however, the solution is only in $\mathrm{H}^1_0(U)$. Improving the regularity of the solution requires elliptic regularity.*

2. *Moreover, Theorem 5.3.6 introduces a μ into the boundary value problem which is not ideal. To fix this one uses compactness arguments.*

5.3.3 Fredholm Alternative

The Fredholm alternative is another method by which the existence of weak solutions may be established. Developing Fredholm's theory requires the introduction of compact notions on Hilbert spaces.

Definition 5.3.8. *Suppose that H is a Hilbert space with inner product $(\cdot,\cdot)$. Let $(u_n)_{n\in\mathbb{N}} \subseteq H$. Then $(u_n)_{n\in\mathbb{N}}$ converges weakly to $u \in H$, denoted $u_n \rightharpoonup u$, if*

$$(u_n, w) \to (u, w)$$

for every $w \in H$.

Remark 5.3.9.

1. *As H^* can be identified with $\{(\cdot, w) : w \in H\}$, we see the resemblances of weak convergence in Hilbert spaces to notions of weak convergence encountered previously.*
2. *Strong convergence implies weak convergence.*

Lemma 5.3.10. *Weak limits are unique when they exist.*

Proof. Suppose for $(u_n)_{n\in\mathbb{N}} \subseteq H$ we have $u_n \to u$ and $u_n \to u'$. Then for any $w \in H$ we have

$$(w, u') - (w, u) = \lim_{n\to\infty} (w, u_n) - \lim_{n\to\infty} (w, u_n) = 0.$$

Therefore, $(w, u' - u) = 0$ for all $w \in H$ which implies that $u' = u$ due to the non-degeneracy of the inner product. □

Theorem 5.3.11. *Let H be a separable Hilbert space, and suppose that $(u_n)_{n\in\mathbb{N}} \subseteq H$ is a bounded sequence, that is*

$$\|u_n\| \leq K$$

for all $n \in \mathbb{N}$. Then $(u_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence. That is, there exists $\left(u_{m_j}\right)_{j\in\mathbb{N}} \subseteq (u_n)_{n\in\mathbb{N}}$ such that $u_{m_j} \rightharpoonup u$ for some $u \in H$ with $\|u\| \leq K$.

Proof. Since H is separable there exists an orthonormal basis $(e_i)_{i=1}^{\infty} \subseteq H$. Consider the sequence $((e_1, u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$. By Cauchy-Schwartz we have

$$|(e_1, u_n)| \leq \|e_1\|\|u_n\| \leq K,$$

that is $((e_1, u_n))_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Therefore, by the Bolzano-Weierstrass theorem, there exists a convergent subsequence $(e_1, u_{m_n}) \to c_1$ for some $|c_1| \leq K$. Consider the sequence $(u_{1,n})_{n\in\mathbb{N}}$ where $u_{1,n} := u_{m_n}$ for $n \in \mathbb{N}$. Replicating the argument for the sequence $(e_2, u_{1,n})_{n\in\mathbb{N}} \subseteq \mathbb{R}$ yields a subsequence $(e_2, u_{1,m_n})_{n\in\mathbb{N}}$ such that $(e_2, u_{1,m_n}) \to c_2$ with $|c_2| \leq K$. In particular, as $(e_2, u_{1,m_n})_{n\in\mathbb{N}} \subseteq (e_1, u_{m_n})_{n\in\mathbb{N}}$ we also have that $(e_2, u_{1,m_n}) \to c_1$. Then set $(u_{2,n})_{n\in\mathbb{N}}$ with $u_{2,n} = (e_2, u_{m_n})$ for $n \in \mathbb{N}$. Proceeding inductively, for all $l \in \mathbb{N}$ we construct a subsequence $(u_{l,n})_{n\in\mathbb{N}}$ such that for each $j = 1, \ldots, l$ we have that

$$(e_j, u_{l,k}) \overset{k\to\infty}{\longrightarrow} c_j$$

for some c_j with $|c_j| \leq K$. Now take the diagonal $(v_l)_{l\in\mathbb{N}}$ where $v_l := u_{l,l}$. Note that $(v_l)_{l\in\mathbb{N}} \subseteq (u_n)_{n\in\mathbb{N}}$ and

$$(e_j, v_l) \overset{l\to\infty}{\longrightarrow} c_j$$

for all $j \in \mathbb{N}$. Observe that

$$\begin{aligned}\sum_{j=1}^{p} |c_j|^2 &= \sum_{j=1}^{p} \lim_{l\to\infty} |(e_j, v_l)|^2 \\ &\overset{(1)}{=} \lim_{l\to\infty} \sum_{j=1}^{p} |(e_j, v_l)|^2 \\ &\leq \sup_{l} \sum_{j=1}^{p} |(e_j, v_l)|^2 \\ &\overset{(2)}{\leq} \sup_{l} \|v_l\|^2 \\ &\leq K^2\end{aligned}$$

where (1) is justified as the sum is finite and (2) is an application of Bessel's inequality. Now taking $p \to \infty$ it follows that

$$\sum_{j=1}^{\infty} |c_j|^2 \leq K^2 < \infty.$$

This implies that $u := \sum_i c_i e_i$ converges in H as its partial sums are Cauchy and H is complete. In particular, we have

$$\|u\| \leq K.$$

Moreover, as $(e_j, v_l) \to c_j = (e_j, u)$ for all j, the weak convergence conditions holds on an orthonormal basis. Thus, fix $w \in H$ and write

$$w = \sum_{i=1}^{p} (e_i, w) e_i + w_p.$$

Then as $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis we know that

$$\sum_{i=1}^{p} (e_i, w) e_i \to w,$$

that is $w_p \to 0$ in H as $\|w\| < \infty$. Note that since $w - w_p$ is a finite linear combination of the e_i's and we know that $(e_j, v_l - u) \to 0$ as $l \to \infty$, we can choose L large enough such that

$$|(w - w_p, v_l - u)| < \frac{\epsilon}{2} \tag{5.3.8}$$

for all $l \geq L$. Similarly,

$$\begin{aligned}|(w_p, v_l - u)| &\leq \|w_p\| \|v_l - u\| \\ &\leq \|w_p\| \left(\|u\| + \|v_l\|\right) \\ &\leq 2K\|w_p\| \\ &\overset{p\to\infty}{\longrightarrow} 0.\end{aligned}$$

Thus, we can choose a p such that

$$|(w_p, v_l - u)| < \frac{\epsilon}{2}. \tag{5.3.9}$$

Therefore, for $l \geq L$ and p sufficiently large we have

$$\begin{aligned}|(w, v_l) - (w, u)| &\leq |(w - w_p, v_l - u)| + |(w_p, v_l - u)| \\ &\overset{\text{(5.3.8)(5.3.9)}}{\leq} \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Hence, we can conclude that $v_l \rightharpoonup u$ in H. □

Proposition 5.3.12. *Suppose $u \in \mathrm{H}^1\left(\mathbb{R}^d\right)$. Let $Q = \prod_{i=1}^d [\xi_i, \xi_i + L]$ be a box of side length L. Then*

$$\|u\|_{L^2(Q)} \leq \frac{1}{|Q|}\left(\int_Q u \,\mathrm{d}x\right)^2 + \frac{dL^2}{2}\|Du\|_{L^2(Q)}^2.$$

Proof. Through approximation arguments it suffices to consider $u \in \mathcal{C}^\infty\left(\bar{Q}\right)$. Let $x, y \in Q$ and observe that

$$\begin{aligned} u(x) - u(y) = &\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \ldots, x_d)\,\mathrm{d}t + \int_{y_2}^{x_2} \frac{\mathrm{d}}{\mathrm{d}t} u(y_1, t, x_3, \ldots, x_d)\,\mathrm{d}t \\ &+ \cdots + \int_{y_d}^{x_d} \frac{\mathrm{d}}{\mathrm{d}t} u(y_1, y_2, \ldots, y_{d-1}, t)\,\mathrm{d}t. \end{aligned}$$

Squaring this expression and applying Cauchy-Schwartz it follows that

$$\begin{aligned} u(x)^2 + u(y)^2 - 2u(x)u(y) \leq d\Bigg(&\left(\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \ldots, x_d)\,\mathrm{d}t\right)^2 \\ &+ \cdots + \left(\int_{y_d}^{x_d} \frac{\mathrm{d}}{\mathrm{d}t} u(y_1, \ldots, y_{d-1}, t)\,\mathrm{d}t\right)^2\Bigg). \end{aligned} \tag{5.3.10}$$

Integrating the left-hand side of (5.3.10) over Q yields

$$\int_Q \int_Q u(x)^2 + u(y)^2 - 2u(x)u(y)\,\mathrm{d}x\,\mathrm{d}y = 2|Q|\|u\|_{L^2(Q)}^2 - 2\left(\int_Q u(x)\,\mathrm{d}x\right)^2.$$

Letting $I_1 := \left(\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \ldots, x_d)\,\mathrm{d}t\right)^2$ we can use Cauchy-Schwartz to write

$$\begin{aligned} I_1 &\leq |y_1 - x_1| \int_{y_1}^{x_1} \left(\frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \ldots, x_d\right)^2 \mathrm{d}t \\ &\leq L \int_{\xi_1}^{\xi_1 + L} \left(\frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \ldots, x_d)\right)^2 \mathrm{d}t. \end{aligned}$$

Which is a bound on I_1 independent of x_1 and y, meaning

$$\int_Q \int_Q I_1 \,\mathrm{d}x\,\mathrm{d}y \leq L^2 |Q| \|D_1 u\|_{L^2(Q)}^2,$$

where the extra L factor comes from the independence from x_1 and the $|Q|$ comes from the independence from y. Bounding each $I_i := \left(\int_{y_k}^{x_k} \frac{\mathrm{d}}{\mathrm{d}t} u(y_1, \ldots, t, \ldots, x_d)\,\mathrm{d}t\right)^2$ in similar ways, then

$$dL^2|Q| \sum_{i=1}^d \|D_i u\|_{L^2(Q)}^2 = dL^2|Q|\|Du\|_{L^2(Q)}^2$$

is a bound on the integral over Q of the right-hand side of (5.3.10). Therefore,

$$2|Q|\|u\|_{L^2(Q)}^2 - 2\left(\int_Q u(x)\,\mathrm{d}x\right)^2 \leq dL^2|Q|\|Du\|_{L^2(Q)}^2$$

which completes the proof. □

Remark 5.3.13. *Suppose ζ is some smooth function in $\mathrm{H}^1\left(\mathbb{R}^d\right)$ which vanishes outside a compact set containing Q and is 1 on Q. With $c = \frac{1}{|Q|}\int_Q u \,\mathrm{d}x$ and applying Proposition 5.3.12 to $u - c\zeta$ it follows that*

$$\|u - c\|_{L^2(Q)}^2 \leq \frac{dL^2}{2}\|Du\|_{L^2(Q)}^2.$$

That is, we can use Proposition 5.3.12 to bound the difference between u and its average with its derivative.

Definition 5.3.14. *Let X and Y be normed vector spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ respectively and $X \subseteq Y$. Then X is compactly embedded in Y, denoted $X \underset{c}{\hookrightarrow} Y$, if the following hold.*

1. *X is continuously embedded in Y. That is, there exists a constant $c > 0$ such that*

$$\|u\|_Y \leq c\|u\|_X$$

   for all $u \in X$.

2. *The embedding is a compact operator. Namely, the embedding of every bounded sequence admits a strongly convergent subsequence in $\|\cdot\|_Y$.*

Theorem 5.3.15 (Rellich-Kondrachov)**.** *Let $U \subseteq \mathbb{R}^d$ be bounded with $\mathcal{C}^1$-boundary. Let $(u_m)_{m\in\mathbb{N}} \subseteq \mathrm{H}^1(U)$ be a bounded sequence with*

$$\|u_m\|_{\mathrm{H}^1(U)} \leq K$$

for all $m \in \mathbb{N}$. Then there exists $u \in \mathrm{H}^1(U)$ and a subsequence $\left(u_{m_j}\right)_{j\in\mathbb{N}}$ such that $u_{m_j} \to u$ in $\mathrm{H}^1(U)$ and $L^2(U)$.

Proof. By Theorem 4.5.2, it suffices to consider $u_m \in \mathrm{H}_0^1(Q)$ for some large cube Q such that $U \Subset Q$. Note that $\mathrm{H}_0^1(U)$ is separable as $L^p(U)$ is separable for all $p < \infty$ which implies that $\mathrm{W}^{k,p}(U)$ is separable for all $p < \infty$. Therefore, by Theorem 5.3.11 there exists a $u \in \mathrm{H}_0^1(U)$ and subsequence $\left(u_{m_j}\right)_{j\in\mathbb{N}}$ such that $u_{m_j} \rightharpoonup u$. Set $u_{m_j} = w_j$ for each $j \in \mathbb{N}$ and fix $\delta > 0$. Note that Q can be covered by taking $k = k(\delta)$ cubes of side-length $L < \delta$ such that the cubes only intersect at their boundaries. Denote this cover of cubes by $\{Q_l\}_{l=1}^k$. Using Proposition 5.3.12 we have

$$\|w_j - u\|_{L^2(Q_l)}^2 \leq \frac{1}{|Q_l|}\left(\int_{Q_l}(w_j - u)(x)\,\mathrm{d}x\right)^2 + \frac{d\delta^2}{2}\|D(w_j - u)\|_{L^2(Q_l)}^2.$$

Summing over l it follows that

$$\begin{aligned}\|w_j - u\|_{L^2(Q)} &= \sum_{l=1}^k \|w_j - u\|_{L^2(Q_l)} \\ &\leq \sum_{l=1}^k \frac{1}{|Q_l|}\left(\int_{Q_l}(w_j - u)(x)\,\mathrm{d}x\right)^2 + \frac{d\delta^2}{2}\|D(w_j - u)\|_{L^2(Q)}^2. \end{aligned} \tag{5.3.11}$$

As $u, w_j \in \mathrm{H}_0^1(U)$ we have

$$\|Dw_j - Du\|_{L^2(Q)}^2 \leq K.$$

Hence, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\frac{d\delta^2}{2}\|D(w_i - u)\|_{L^2(Q)}^2 < \frac{\epsilon}{2} \tag{5.3.12}$$

for all $j \in \mathbb{N}$. Moreover, since the map $u \mapsto \int_Q u(x)\,\mathrm{d}x$ is linear and bounded in $\mathrm{H}^1(Q)$, the weak convergence of $(w_j)_{j\in\mathbb{N}}$ to u implies that

$$\int_{Q_l} (w_j - u)(x)\,\mathrm{d}x \overset{j\to\infty}{\longrightarrow} 0$$

for each $1 \leq l \leq k$. In particular, j can be chosen large enough such that

$$\sum_{l=1}^{k} \frac{1}{|Q_l|} \left(\int_{Q_l} (w_j - u)(x)\,\mathrm{d}x \right)^2 < \frac{\epsilon}{2}. \tag{5.3.13}$$

Returning to (5.3.11) with (5.3.12) and (5.3.13) it follows that

$$\|w_j - u\| < \epsilon.$$

□

Remark 5.3.16.

1. *In other words, Theorem 5.3.15 says that $\mathrm{H}^1(U) \underset{c}{\hookrightarrow} L^2(U)$. Consequently, we can improve the regularity of the weak solutions provided by Theorem 5.3.2.*
2. *A similar result holds for $\mathrm{H}^1_0(U)$ that does not require the assumption on the boundary.*
3. *More generally, one can show that $\mathrm{W}^{1,p} \underset{c}{\hookrightarrow} L^p$ for $p \in [1,\infty)$.*

Definition 5.3.17. *Let H be a Hilbert space. Then a bounded operator $K : H \to H$ is compact if for each bounded sequence $(u_n)_{n\in\mathbb{N}} \subseteq H$, there exists a subsequence $\left(K\left(u_{n_j}\right)\right)_{j\in\mathbb{N}}$ which converges strongly in H.*

Remark 5.3.18. *Compact operators are a generalisation of finite rank operators, such as matrices.*

Example 5.3.19.

1. *Let $K : L^2(U) \to \mathrm{H}^1(U)$ be a bounded linear operator. We know that $\mathrm{H}^1(U) \subseteq L^2(U)$ and so we can view K as an operator $K : L^2(U) \to L^2(U)$. Let $(u_n)_{n\in\mathbb{N}} \subseteq L^2(U)$ be a bounded sequence. Then since K is bounded we have*
$$\|K(u_n)\|_{\mathrm{H}^1(U)} \leq \|K\|\|u_n\|_{L^2(U)} \leq c.$$
Using Theorem 5.3.15 we can extract a subsequence $\left(u_{n_j}\right)_{j\in\mathbb{N}}$ such that $K\left(u_{n_j}\right) \to K(u)$ in $L^2(U)$. Thus, $K : L^2(U) \to L^2(U)$, and in particular $K : L^2(U) \to \mathrm{H}^1(U)$ is compact.
2. *A k^{th} order elliptic problem can be formulated as*
$$Lu = f \tag{5.3.14}$$
where $f \in L^2(U)$ and $L : \mathrm{H}^{k-1}(U) \to L^2(U)$. Focusing on the case when $k = 2$ we have $L : \mathrm{H}^1(U) \to L^2(U)$. To solve (5.3.14) *one can find an inverse map $L^{-1} : L^2(U) \to \mathrm{H}^1(U)$. From statement 1 of this example, one can apply Theorem 5.3.15 to deduce that L^{-1} is a compact operator. In particular, we will see that this will let us apply Theorem 5.3.22 to deduce information about the solutions of* (5.3.14).

Exercise 5.3.20. *Let K be a compact linear operator, and let T be a bounded linear operator. Show that $K \circ T$ is a compact linear operator.*

Lemma 5.3.21. *Let H be a Hilbert space. Then the identity operator $I : H \to H$ is a compact operator if and only if H is finite-dimensional.*

Proof. ($\Leftarrow$). Let $(u_n)_{n\in\mathbb{N}} \subseteq H$ be a bounded sequence. Then by Theorem 5.3.11 there exists a weakly convergent subsequence. Since strong convergence is equivalent to weak convergence in finite dimensions, it follows that this subsequence converges strongly. Therefore, I is compact.
($\Rightarrow$). In infinite dimensions, the closed unit ball in H is not compact. Therefore, $I : H \to H$ cannot be compact. Therefore, if $I : H \to H$ is compact it must be the case that $\dim(H) < \infty$. □

Theorem 5.3.22 (Fredholm Alternative)**.** *Let H be a Hilbert space and $K : H \to H$ a real compact operator. Then the following hold.*

1. $\ker(I - K)$ *is finite-dimensional.*
2. $\operatorname{im}(I - K)$ *is closed.*
3. $\operatorname{im}(I - K) = \ker(I - K^*)^{\perp}$.
4. $\ker(I - K) = \{0\}$ *if and only if* $\operatorname{im}(I - K) = H$.
5. $\dim(\ker(I - K)) = \dim((I - K^*)) < \infty$.

Here $I : H \to H$ is the identity operator and K^* is the adjoint operator of K.

Proof. See [3]. □

Remark 5.3.23. *From statement 4 of Theorem 5.3.22 we see that either*

1. $(I - K)u = u - Ku = f$ *has a unique solution for all $f \in H$, or*
2. $(I - K)u = 0$ *has solutions non-zero solutions.*

Case 1 is the inhomogeneous case and case 2 is the homogeneous case. In the homogeneous case, the space of solutions is finite-dimensional by statement 1 of Theorem 5.3.22. Furthermore, in case 2 the inhomogeneous formulation has a solution if and only if $f \in \ker(I - K^*)^{\perp}$ by statement 3 of Theorem 5.3.22.

Consider

$$Lu = f$$

where

$$Lu = -\sum_{i,j=1}^{d} \left(a_{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{d} b_i(x)u_{x_i} + c(x)u.$$

Assume that L is uniformly elliptic on U, which is an open bounded set with $\mathcal{C}^1$-boundary. The associated bilinear form is given by

$$B[u, v] = \int_U \sum_{i,j=1}^{d} a_{ij}(x)u_{x_i}v_{x_j}\,\mathrm{d}x + \int_U \sum_{i=1}^{d} b_i(x)u_{x_i}v\,\mathrm{d}x + \int_U c(x)u(x)v(x)\,\mathrm{d}x.$$

We would like to understand the boundary value problem

$$\begin{cases} Lu = f & U \\ u = 0 & \partial U. \end{cases}$$

To do so we consider the formal adjoint of L given by

$$L^*v := -\sum_{i,j=1}^{d}\left(a_{ij}v_{x_j}\right)_{x_i} - \sum_{i=1}^{d} b_i v_{x_i} + \left(c - \sum_{i=1}^{d}(b_i)_{x_i}\right)v.$$

provided $b_i \in \mathcal{C}^1\left(\bar{U}\right)$.

Exercise 5.3.24. *Show that for $\varphi, \psi \in \mathcal{C}_c^\infty(U)$ we have*

$$(L\varphi, \psi)_{L^2(U)} = (\varphi, L^*\psi)_{L^2(U)}.$$

Remark 5.3.25. *From Exercise 5.3.24 it follows through density arguments that*

$$(Lu, v)_{L^2(U)} = (u, L^*v)_{L^2(U)}$$

for every $u, v \in \mathrm{H}_0^1(U)$.

Let the adjoint bilinear form $B^*[\cdot,\cdot]$ be given by

$$B^*[v,u] := B[u,v]. \tag{5.3.15}$$

Note that when $b_i \in \mathcal{C}^1\left(\bar{U}\right)$ the bilinear form B^* can be equivalently defined as the bilinear form corresponding to L^*. Indeed,

$$\begin{aligned} B[u,v] &= (Lu, v)_{L^2(U)} \\ &= (u, L^*v)_{L^2(U)} \\ &= (L^*v, u)_{L^2(U)} \\ &= B^*[v,u]. \end{aligned}$$

However, the defining equation (5.3.15) does not require L^* and thus makes sense even when $b_i \in L^\infty(U)$.

Definition 5.3.26. *A function $v \in \mathrm{H}_0^1(U)$ is a weak solution to the adjoint problem*

$$\begin{cases} L^*v = f & U \\ v = 0 & \partial U \end{cases}$$

if

$$B^*[\cdot, v] = (f, \cdot)_{L^2(U)}$$

as maps on $\mathrm{H}_0^1(U)$.

Exercise 5.3.27. *As usual, consider $a_i, b_i, c \in L^\infty(U)$ with the a_{ij} satisfying the uniform ellipticity condition*

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $\xi \in \mathbb{R}^d$ and some $\theta > 0$ for almost every $x \in U$. For $\gamma > 0$ sufficiently large, let $L_\gamma^{-1} : L^2(U) \to \mathrm{H}_0^1(U)$ be the bounded linear operator which maps $f \in L^2(U)$ to the weak solution $u \in \mathrm{H}_0^1(U)$ of the problem

$$\begin{cases} Lu + \gamma u = f & U \\ u = 0 & \partial U. \end{cases}$$

Show that $\operatorname{im}\left(L_\gamma^{-1}\right)$ *is dense in* $\mathrm{H}_0^1(U)$.

Theorem 5.3.28. *For* U *an open and bounded set with* $\mathcal{C}^1$*-boundary, consider the boundary value problem*

$$\begin{cases} Lu = f & U \\ u = 0 & \partial U \end{cases} \tag{5.3.16}$$

for L *a uniformly elliptic operator. Then exactly one of the following holds.*

1. *For all* $f \in L^2(U)$ *there exists a unique solution* $u \in \mathrm{H}_0^1(U)$ *to* (5.3.16).
2. *There exists a non-zero weak solution* $u \in \mathrm{H}_0^1(U)$ *to the homogeneous formulation of* (5.3.16)*, that is* $f = 0$.

In particular, if statement 2 holds then the dimension of $N \subseteq \mathrm{H}_0^1(U)$ *the subspace of weak solutions to homogeneous formulation of* (5.3.16) *equals the dimension of* $N^* \subseteq \mathrm{H}_0^1(U)$ *the subspace of weak solutions to the adjoint formulation of the homogeneous formulation of* (5.3.16)*. More specifically,* $\dim(N) = \dim(N^*) < \infty$*. Furthermore, the inhomogeneous formulation of* (5.3.16) *has a unique weak solution if and only if*

$$(f, v)_{L^2(U)} = 0$$

for all $v \in N^*$.

Proof. Step 1: Apply Theorem 5.3.4.
Let

$$B_\gamma[u, v] := B[u, v] + \gamma(u, v)$$

be the bilinear form corresponding to $L_\gamma u := Lu + \gamma u$, where γ comes from Theorem 5.3.4. Then Theorem 5.3.6 tells us that for each $g \in L^2(U)$ there exists a unique function $u \in \mathrm{H}_0^1(U)$ that solves

$$B_\gamma[u, v] = (g, v) \tag{5.3.17}$$

for all $v \in \mathrm{H}_0^1(U)$. Let us write $u = L_\gamma^{-1} g$ to indicate when (5.3.17) holds.
Step 2: Identify an equivalent condition for the solutions of (5.3.16).
Note that $u \in \mathrm{H}_0^1(U)$ is a solution to (5.3.16) if and only if $B[u, v] = (f, v)_{L^2(U)}$ for all $v \in \mathrm{H}_0^1(U)$. Hence,

$$B_\gamma[u, v] = B[u, v] + \gamma(u, v)_{L^2(U)} = (f + \gamma u, v)_{L^2(U)}.$$

Thus, a u is a weak solution to (5.3.16) if and only if $u = L_\gamma^{-1}(\gamma u + f)$. Which we can equivalently write as

$$u - Ku = h$$

for $Ku := \gamma L_\gamma^{-1} u$ and $h := L_\gamma^{-1} f$.
Step 3: Show that K is a compact operator.
Note that from the choice of γ

$$\begin{aligned} \beta \|u\|_{\mathrm{H}_0^1(U)}^2 &\overset{\text{Thm 5.3.4}}{\leq} B_\gamma[u, u] \\ &\overset{(5.3.17)}{=} (g, u)_{L^2(U)} \\ &\leq \|g\|_{L^2(U)} \|u\|_{L^2(U)} \\ &\leq \|g\|_{L^2(U)} \|u\|_{\mathrm{H}_0^1(U)}. \end{aligned}$$

Hence,

$$\begin{aligned} \|Kg\|_{\mathrm{H}_0^1(U)} &= \left\|\gamma L_\gamma^{-1} g\right\|_{\mathrm{H}_0^1(U)} \\ &= \gamma \|u\|_{\mathrm{H}_0^1(U)} \\ &\leq \frac{\gamma}{\beta} \|g\|_{L^2(U)}. \end{aligned}$$

Therefore, $K : L^2(U) \to \mathrm{H}_0^1(U)$ is a bounded linear operator and thus compact by the same reasoning of statement 1 of Example 5.3.19.

Step 4: Apply Theorem 5.3.22.

Applying Theorem 5.3.22 it follows that one of the following hold.

(α). For all $h \in L^2(U)$ there exists a unique solution $u \in \mathrm{H}_0^1(U)$ to the equation $u - Ku = h$.

(β). The equation $u - Ku = 0$ has non-zero solutions in $\mathrm{H}_0^1(U)$.

Step 5: Understand the different cases from step 4.

If (α) holds, then we can take $h = L_\gamma^{-1}(f)$ to note from step 2 that there exists a unique solution to (5.3.16). On the other hand, if (β) holds, then $u - Ku = 0$, thus,

$$B[u,v] + \gamma(u,v)_{L^2(U)} = (\gamma u, v)_{L^2(U)}$$

for all $v \in \mathrm{H}_0^1(U)$. That is, $B[u,v] = 0$ for all $v \in \mathrm{H}_0^1(U)$, or in other words $u \in \mathrm{H}_0^1(U)$ solves the homogeneous formulation of (5.3.16).

Step 6: Show that $v - K^*v = 0$ if and only if $v \in N^*$.

Note that $v - K^*v = 0$ if and only if $v = K^*v$ which happens if and only if

$$(v,w)_{L^2(U)} = (K^*v, w)_{L^2(U)} = (v, Kw)_{L^2(U)}$$

for all $w \in L^2(U)$. Which is equivalent to

$$(v,w)_{L^2(U)} = \left(v, \gamma L_\gamma^{-1} w\right)_{L^2(U)} \tag{5.3.18}$$

for all $w \in L^2(U)$. Recall that a weak solution to

$$\begin{cases} L_\gamma \varphi = \tilde{f} & U \\ \varphi = 0 & \partial U \end{cases}$$

where $\tilde{f} \in L^2(U)$, satisfies

$$B[\varphi, v] + \gamma(\varphi, v)_{L^2(U)} = \left(\tilde{f}, \varphi\right)_{L^2(U)}.$$

Letting $\tilde{f} = w$ we get that $\varphi = L_\gamma^{-1} w$ and so

$$B\left[L_\gamma^{-1}(w), v\right] + \gamma\left(L_\gamma^{-1}(w), v\right)_{L^2(U)} = (w,v)_{L^2(U)}. \tag{5.3.19}$$

Using (5.3.18) and (5.3.19) we deduce that $v - K^*v = 0$ if and only if

$$\left(v, \gamma L_\gamma^{-1}(w)\right)_{L^2(U)} = B\left[L_\gamma^{-1}(w), v\right] + \gamma\left(L_\gamma^{-1}(w), v\right)_{L^2(U)}$$

for all $w \in L^2(U)$. Which is equivalent to $B\left[L_\gamma^{-1}(w), v\right] = 0$ and thus $B^*\left[v, L_\gamma^{-1}(w)\right] = 0$ for all $w \in L^2(U)$. However, we also know that v is a solution to the homogeneous adjoint formulation of (5.3.16) if and only if $B^*[v,u] = 0$ for all $u \in \mathrm{H}_0^1(U)$. But from Exercise 5.3.27 we know that $\operatorname{im}\left(L_\gamma^{-1}\right) \subseteq \mathrm{H}_0^1(U)$ is dense. Thus, since L_γ^{-1} is a continuous operator we get that $B^*\left[v, L_\gamma^{-1}(w)\right] = 0$ for all $w \in L^2(U)$ if and only if $B^*[v,u] = 0$ for all $u \in \mathrm{H}_0^1(U)$. Therefore, v is a weak solution to the homogeneous adjoint problem of (5.3.16), that is $v \in N^*$. In particular, $\dim(N^*) = \dim(\ker(I - K^*))$. Similarly, $\dim(N) = \dim(\ker(I-K))$. Thus, from statement 5 of Theorem 5.3.22 we have that $\dim(N) = \dim(N^*) < \infty$

Step 7: Show that (5.3.16) has a weak solution if and only if $(f,v)_{L^2(U)} = 0$ for all $v \in N^*$.

Note that (5.3.16) has a solution if and only if $(I-K)u = L_\gamma^{-1}(f)$, that is, if and only if $L_\gamma^{-1}(f) \in \operatorname{im}(I-K)$. Using statement 3 of Theorem 5.3.22 this is equivalent to $L_\gamma^{-1}(f) \in \ker(I-K^*)^\perp$ which is in turn equivalent to

$$\left(v, L_\gamma^{-1}(f)\right)_{L^2(U)} = 0$$

for all $v \in \ker(I - K^*)$. Observe that

$$\begin{aligned}\left(v, L_\gamma^{-1}(f)\right)_{L^2(U)} &= \left(v, \frac{1}{\gamma}Kf\right)_{L^2(U)} \\ &= \frac{1}{\gamma}(v, Kf)_{L^2(U)} \\ &= \frac{1}{\gamma}\left(K^*v, f\right)_{L^2(U)} \\ &= \frac{1}{\gamma}(v, f)_{L^2(U)}\end{aligned}$$

where the last equality follows for $v \in \ker(I - K^*)$. Hence, we see that $\left(v, L_\gamma^{-1}f\right)_{L^2(U)} = 0$ for all $v \in \ker(I - K^*)$ if and only if $(v, f)_{L^2(U)} = 0$ for all $v \in \ker(I - K^*)$. From step 6 we know that $v \in \ker(I - K^*) = N^*$, and thus we are done. ☐

Remark 5.3.29. *Theorem 5.3.28 is an analogous result to the case of the matrix equation $Ax = b$, where either there exists a unique solution or $\ker(A) \neq 0$.*

5.3.4 Spectral Theory

Definition 5.3.30. *For a real Hilbert space let $T : H \to H$ be a bounded linear operator. Then the resolvent of T is*

$$\rho(T) := \{\eta \in \mathbb{R} : (T - \eta I) \text{ is bijective}\}.$$

The real spectrum of T is $\sigma(T) := \mathbb{R} \setminus \rho(T)$.

Remark 5.3.31. *If $\eta \in \rho(T)$ then the closed graph theorem implies that $(T - \eta I)^{-1} : H \to H$ is a bounded linear operator.*

Definition 5.3.32. *A number $\mu \in \sigma(T)$ belongs to the point spectrum of T, denoted $\sigma_p(T)$, if*

$$\ker(T - \mu I) \neq \{0\}.$$

In such a case, $w \in \ker(T - \mu I) \setminus \{0\}$ is an associated eigenvector.

Remark 5.3.33. *The point spectrum consists of the eigenvalues of T.*

Lemma 5.3.34. *Let H be a Hilbert space. Then for a linear, bounded and self-adjoint operator $K : H \to H$ we have that $\sigma(K) \subseteq [m, M]$ with $m, M \in \sigma(K)$ where*

$$m := \inf_{u \in H, \|u\|=1} (Ku, u)$$

and

$$M := \sup_{u \in H, \|u\|=1} (Ku, u).$$

Proof. Let $\eta > M$. Then,

$$(\eta u - Ku, u) \geq (\eta - M)\|u\|^2$$

for $u \in H$. Therefore, using Theorem 5.3.2 it follows that $\eta I - K$ is bijective and thus $\eta \in \rho(K)$. Similarly, if $\eta < m$ we have that $\eta \in \rho(K)$ and so $\sigma(K) \subseteq [m, M]$. Note that $[u, v] := (Mu - Ku, v)$ is self-adjoint with $[u, u] \geq 0$ for all $u \in H$. So from the Cauchy-Schwartz inequality, we have that

$$\begin{aligned}|(Mu - Ku, v)| &= |[u, v]| \\ &\leq \sqrt{[u, u]}\sqrt{[v, v]} \\ &= \sqrt{(Mu - Ku, u)}\sqrt{(Mv - Kv, v)},\end{aligned}$$

for all $u, v \in H$. In particular,

$$\|Mu - Ku\| \leq C\sqrt{(Mu - Ku, u)} \tag{5.3.20}$$

for all $u \in H$. Let $(u_k)_{k\in\mathbb{N}} \subseteq H$ be such that $\|u_k\| = 1$ and $(Ku_k, u_k) \to M$. Then from (5.3.20) it follows that $\|Mu_k - Ku_k\| \to 0$. Therefore, if $M \in \rho(K)$ we would have

$$u_k = (MI - K)^{-1}(Mu_k - Ku_k) \to 0,$$

which is a contradiction as $\|u_k\| = 1$ for $k \in \mathbb{N}$. Therefore, $M \in \sigma(K)$. Similarly, $m \in \sigma(K)$. □

Theorem 5.3.35. *Suppose that $\dim(H) = \infty$ and $K : H \to H$ is a compact operator. Then the following statements hold.*

1. *$0 \in \sigma(K)$.*
2. *$\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$.*
3. *$\sigma(K) \setminus \{0\}$ is either finite or a sequence converging to zero. In particular, $\sigma(K)$ is countable.*

What's more, if K is self-adjoint, and H is separable, then there exists a countable orthonormal basis of H consisting of eigenvectors of K.

Proof.

1. Suppose $0 \notin \sigma(K)$. Then $K : H \to H$ is bijective. In particular, $K \circ K^{-1} = I$ is compact by Exercise 5.3.20. However, this contradicts Lemma 5.3.21.

2. Let $\eta \in \sigma(K) \setminus \{0\}$. Suppose $\ker(K - \eta I) = \{0\}$ then it follows from statement 4 Theorem 5.3.22 that $\operatorname{im}(K - \eta I) = H$. Thus, $K - \eta I$ is bijective meaning $\eta \in \rho(K)$. However, this is a contradiction since $\rho(K) = \mathbb{R} \setminus \sigma(K)$. Therefore, $\ker(K - \eta I) \neq \{0\}$ which implies that $\eta \in \sigma_p(K)$ and so $\sigma(K) \setminus \{0\} \subseteq \sigma_p(K) \setminus \{0\}$. On the other hand, it is clear from Definition 5.3.32 that $\sigma_p(K) \setminus \{0\} \subseteq \sigma(K) \setminus \{0\}$ and so we conclude that $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$.

3. Let $(\eta_k)_{k\in\mathbb{N}} \subseteq \sigma(K)\setminus\{0\}$ be a sequence of distinct elements. Since K is compact this sequence is bounded, by Lemma 5.3.34, and thus contains a convergent subsequence. Suppose without loss of generality that $\eta_k \to \eta$ as $k \to \infty$. Moreover, for contradiction, suppose that $\eta \neq 0$. Since $\eta_k \in \sigma_p(K) \setminus \{0\}$ there exists a $w_k \in H \setminus \{0\}$ such that
$$Kw_k = \eta_k w_k.$$
Let
$$H_k := \operatorname{span}\left(\{w_1, \ldots, w_k\}\right).$$
Since the w_k are non-zero and linearly independent we have that $H_k \subseteq H_{k+1}$ is a strict inclusion for $k \geq 1$. Moreover, for w_i with $i \leq k - 1$ we
$$(K - \eta_k I)w_i = \eta_i w_i - \eta_k w_i \in H_{k-1}$$
and
$$(K - \eta_k I)w_k = \eta_k w_k - \eta_k w_k = 0 \in H_{k-1}.$$

Hence,

$$(K - \eta_k I)H_k \subseteq H_{k-1}.$$

Now let $u_k \in H_k$ so that $u_k \in H_{k-1}^{\perp}$ with $\|u_k\| = 1$. Then for $k > l$ we have

$$H_{l-1} \subseteq H_l \subseteq H_{k-1} \subseteq H_k$$

and so

$$\begin{aligned}
\left\|\frac{1}{\eta_k}Ku_k - \frac{1}{\eta_l}Ku_l\right\| &= \left\|\frac{1}{\eta_k}\underbrace{(Ku_k - \eta_k u_k)}_{\in H_{k-1}} - \frac{1}{\eta_l}\underbrace{(Ku_l - \eta_l u_l)}_{\in H_{l-1}\subseteq H_{k-1}} + \underbrace{u_k}_{\in H_{k-1}^{\perp}} - \underbrace{u_l}_{\in H_{k-1}}\right\| \\
&= \left\|\frac{1}{\eta_k}(Ku_k - \eta_k u_k) - \frac{1}{\eta_l}(K - \eta_l)\, u_l - u_l\right\| + 1 \\
&\geq 1.
\end{aligned}$$

However, if $\eta_k \to \eta \neq 0$ then this contradicts the compactness of K as the sequence $\left(K\left(\frac{1}{\eta_j}u_j\right)\right)_{j\in\mathbb{N}}$ is bounded with no Cauchy, and thus convergent, subsequence. Therefore $\sigma(K) \setminus \{0\}$ is finite or a sequence converging to zero.

Let $(\eta_k)_{k\in\mathbb{N}} \subseteq \sigma(K)\setminus\{0\}$ be the sequence of all distinct eigenvalues, set $\eta_0 = 0$ and consider $H_k := \ker(K - \eta_k I)$. Then, $\dim(H_0) \in [0,\infty]$ and $\dim(H_k) \in (0,\infty)$ by statement 5 of Theorem 5.3.22. Let $u \in H_k$ and $v \in H_l$ for $k \neq l$. Then,

$$\eta_k(u,v) = (Ku,v) = (u,Kv) = \eta_l(u,v),$$

which implies that $(u,v) = 0$ as $\eta_k \neq \eta_l$. Let $\tilde{H}$ be the smallest subspace of H containing $H_0, H_1, \ldots$, that is

$$\tilde{H} = \left\{\sum_{k=0}^{m} a_k u_k : m \in \{0,1,\ldots\},\, u_k \in H_k,\, a_k \in \mathbb{R}\right\}.$$

Note that $K\left(\tilde{H}\right) \subseteq \tilde{H}$. Moreover, if $u \in \tilde{H}^{\perp}$ and $v \in \tilde{H}$, then $(Ku,v) = (u,Kv) = 0$, which implies that $K\left(\tilde{H}^{\perp}\right) \subseteq \tilde{H}^{\perp}$. The operator $\tilde{K} := K|_{\tilde{H}^{\perp}}$ is also compact and self-adjoint. Moreover, $\sigma\left(\tilde{K}\right) = \{0\}$. Then by Lemma 5.3.34 it follows that $\left(\tilde{K}u,u\right) = 0$ for all $u \in \tilde{H}^{\perp}$. However, this means that for $u,v \in \tilde{H}^{\perp}$ we have that

$$2\left(\tilde{K}u,v\right) = \left(\tilde{K}(u+v),u+v\right) - \left(\tilde{K}u,u\right) - \left(\tilde{K}v,v\right) = 0.$$

This, $\tilde{K} = 0$ and so $\tilde{H}^{\perp} \subseteq \ker(K) \subseteq \tilde{H}$, which implies that $\tilde{H}^{\perp} = 0$. Therefore, $\tilde{H}$ is dense in H. We can choose an orthonormal basis for each H_k for $k = 1,2,\ldots$ and since H is separable we can choose a countable orthonormal for H_0 too. Consequently, we obtain an orthogonal basis of eigenvectors for H. ☐

Theorem 5.3.36. *Let U be an open and bounded set with $\mathcal{C}^1$-boundary, and let L be a uniformly elliptic operator.*

1. *Then there exists at most a countable set $\Sigma \subseteq \mathbb{R}$ such that*

$$\begin{cases} Lu = \lambda u + f & U \\ u = 0 & \partial U \end{cases} \tag{5.3.21}$$

   has a unique weak solution for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$.

2. *In particular, if Σ is infinite, then $\Sigma = (\lambda_k)_{k\in\mathbb{N}}$ is a non-decreasing sequence with $\lambda_k \to \infty$.*

3. *For each $\lambda \in \Sigma$, the space $\mathcal{E}(\lambda) \subseteq \mathrm{H}_0^1(U)$ containing weak solutions to the homogeneous formulation of (5.3.21) is finite-dimensional.*

Proof.

1. Let γ be the constant from Theorem 5.3.6, then for $\lambda \leq -\gamma$ the problem

$$\begin{cases} Lu - \lambda u = f & U \\ u = 0 & \partial U \end{cases}$$

has a unique weak solution for all $f \in L^2(U)$ by Theorem 5.3.6. Therefore, $\Sigma \subseteq \{\lambda > -\gamma\}$. Hence, assume that $\lambda > -\gamma$ and consider without loss of generality that $\gamma > 0$. Then as noted in Remark 5.3.23 we know that (5.3.21) has a unique solution for each $f \in L^2(U)$ if and only if $u \equiv 0$ is the only weak solution to the homogeneous formulation of (5.3.21). Equivalently, $u \equiv 0$ is the only weak solution to

$$\begin{cases} Lu + \gamma u = (\gamma + \lambda)u & U \\ u = 0 & \partial U. \end{cases}$$

which holds exactly when

$$u = L_\gamma^{-1}(\gamma + \lambda)u = \frac{\gamma + \lambda}{\gamma} Ku, \tag{5.3.22}$$

where $Ku := \gamma L_\gamma^{-1} u$. From the proof of Theorem 5.3.28 we note that $K : L^2(U) \to L^2(U)$ is a bounded, linear and compact operator. Note that if $u \equiv 0$ is the only solution to (5.3.22) it follows that $\frac{\gamma}{\gamma+\lambda}$ is not an eigenvalue of K. Therefore, we deduce that (5.3.21) has a unique weak solution for each $f \in L^2(U)$ if and only if $\frac{\gamma}{\gamma+\lambda}$ is not an eigenvalue of K. In other words, we characterise Σ with $\lambda \in \Sigma$ if and only if $\frac{\gamma}{\lambda+\gamma}$ is an eigenvalue of K.

2. As K is compact we know from Theorem 5.3.35 that the set of eigenvalues of K is finite or a sequence converging to zero. In the latter case, as $\lambda > -\gamma$ and $\gamma > 0$ it follows that (5.3.21) has a unique weak solution for each $f \in L^2(U)$ except for a sequence $(\lambda_k)_{k\in\mathbb{N}}$ where $\lambda_k \to \infty$.

3. From Theorem 5.3.28, we know that $\mathcal{E}(\lambda)$ is finite-dimensional.

$\square$

Remark 5.3.37.

1. *A number $\lambda \in \Sigma$, as given by Theorem 5.3.36, is referred to as an eigenvalue of L with $u \in \mathcal{E}(\lambda)$ being a corresponding eigenvector.*

2. *Statement 1 of Theorem 5.3.36 tells us that the λ for which* (5.3.21) *does not have a weak solution for any f is at most countable. In other words, we can almost always solve* (5.3.21).

Theorem 5.3.38. *Suppose*

$$\begin{cases} Lu = f & U \\ u = 0 & \partial U \end{cases} \tag{5.3.23}$$

has a weak solution $u \in \mathrm{H}_0^1(U)$ for all $f \in L^2(U)$. Then there exists a constant C such that

$$\|u\|_{L^2(U)} \leq C\|f\|_{L^2(U)}$$

for $f \in L^2(U)$ and solutions $u \in \mathrm{H}_0^1(U)$ to (5.3.23).

Proof. Suppose that there exists $(u_n)_{n\in\mathbb{N}} \subseteq \mathrm{H}_0^1(U)$ and $(f_n)_{n\in\mathbb{N}} \subseteq_{L^2(U)}$ such that

$$\begin{cases} Lu_n = f_n & U \\ u_n = 0 & \partial U \end{cases}$$

with $\|u_n\|_{L^2(U)} \geq n\|f_n\|L^2(U)$. Since L is linear, we can assume without loss of generality that $\|u_n\|_{L^2(U)} = 1$ for $n \in \mathbb{N}$. Then $\|f_n\|_{L^2(U)} \leq \frac{1}{n}$ implying that $f_n \to 0$ in $L^2(U)$. Using (5.3.3) we note that

$$\begin{aligned}
\beta\|u_n\|_{\mathrm{H}^1(U)} &\leq B[u_n, u_n] + \gamma\|u_n\|^2_{L^2(U)} \\
&= (f_n, u_n)_{L^2(U)} + \gamma \\
&\overset{\text{Hölder's}}{\leq} \|f_n\|_{L^2(U)}\|u_n\|_{L^2(U)} + \gamma \\
&\leq \frac{1}{n} + \gamma \\
&\leq \gamma + 1.
\end{aligned}$$

Therefore, $(u_n)_{n\in\mathbb{N}} \subseteq \mathrm{H}^1(U)$ is bounded, and so using Theorem 5.3.15 we can extract a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $u_k \to u$ in $L^2(U)$ to some $u \in \mathrm{H}^1(U)$. In particular,

$$\begin{cases} Lu = 0 & U \\ u = 0 & \partial U \end{cases}$$

and thus $u \equiv 0$ since solutions are unique. However, as $\|u_{n_k}\|_{L^2(U)} = 1$ for $k \in \mathbb{N}$ we must have $\|u\|_{L^2(U)} = 1$. Hence, we get a contradiction, thus there exists a constant C such that

$$\|u\|_{L^2(U)} \leq C\|f\|_{L^2(U)}$$

for all $f \in L^2(U)$ and $u \in \mathrm{H}^1_0(U)$ solving

$$\begin{cases} Lu = f & U \\ u = 0 & \partial U. \end{cases}$$

$\square$

5.4 Elliptic Regularity

Our aim is to improve the regularity of our solutions from $\mathrm{H}^1_0(U)$ to say $\mathrm{H}^2(U)$. With $u \in \mathrm{H}^1_0(U)$ it is not clear whether $u \in \mathrm{H}^2(U)$ and so $Lu = f$ does not make sense. Moreover, even if $u \in \mathrm{H}^2(U)$ it does not mean that u is differentiable in the classical sense, thus $Lu = f$ may not hold in the strong sense. However, such conclusions are possible under some assumptions on L.

Example 5.4.1. *Consider the Poisson equation*

$$-\Delta u = f \tag{5.4.1}$$

for $f \in \mathcal{C}^\infty_c\left(\mathbb{R}^d\right)$ and $u \in \mathcal{C}^\infty_c\left(\mathbb{R}^d\right)$. Then

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)^2 \,\mathrm{d}x &= \int_{\mathbb{R}^d} (\Delta u)^2 \,\mathrm{d}x \\
&= \sum_{i,j=1}^d \int_{\mathbb{R}^d} (D_iD_iu)(D_jD_ju) \,\mathrm{d}x \\
&\overset{(1)}{=} \sum_{i,j=1}^d \int_{\mathbb{R}^d} (D_iD_ju)(D_iD_ju) \,\mathrm{d}x \\
&= \sum_{i,j=1}^d \int_{\mathbb{R}^d} |D_iD_ju|^2 \,\mathrm{d}x \\
&= \left\|D^2u\right\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

where (1) follows as u is smooth. Therefore, the L^2-norm of the second derivatives of u can be estimated, *exactly, by the L^2-norm of f. Similarly, from differentiating (5.4.1) we get that*

$$-\Delta u_{x_k} = f_{x_k}$$

for each $k = 1, \ldots, n$. Applying similar reasoning we deduce that the L^2-norm of the third derivatives of u can be estimated by the first derivatives of f. More generally, the L^2-norm of the $(m+2)^{th}$ derivatives of u can be controlled by the L^2-norm of the m^{th} derivatives of f for $m = 0, 1, \ldots$.

Our aim is to extrapolate the ideas of Example 5.4.1 using difference quotients.

Definition 5.4.2. *Let $U \subseteq \mathbb{R}^d$ be open with $V \Subset U$. For $0 < |h| < \operatorname{dist}(V, \partial U)$ let*

$$\Delta_k^h u(x) := \frac{u(x + he_k) - u(x)}{h}$$

for $k = 1, \ldots, d$. Then the difference quotient is

$$\Delta^h u := \left(\Delta_1^h u, \ldots, \Delta_d^h u\right).$$

Remark 5.4.3.

1. *By construction $x + he_i \in U$.*
2. *Suppose $u \in L^2(U)$, then $\Delta^h u \in L^2(V)$.*
3. *Suppose $u \in \mathrm{H}^1(U)$, then $\Delta^h u \in \mathrm{H}^1(V)$ and $D\left(\Delta^h u\right) = \Delta^h(Du)$.*

Lemma 5.4.4. *Let $u, v \in \mathrm{H}_0^1(U)$ be compactly supported in $V \Subset U$. Then for sufficiently small h the following statements hold.*

1. *$\int_V u\left(\Delta_k^h v\right) \mathrm{d}x = -\int_V \left(\Delta_k^{-h} u\right) v \, \mathrm{d}x$.*
2. *$\Delta_k^h(vu) = \left(\tau_k^h v\right) \Delta_k^h u + \left(\Delta_k^h v\right) u$ where $\tau_k^h v(x) := v(x + he_k)$ is the translation operator.*

Proof.

1. Suppose $v \in \mathcal{C}_c^\infty(U)$. Since V is open and v has compact, there exists $0 < 2|h| < \operatorname{dist}(\operatorname{supp}(v), \partial V)$. Therefore,

$$\begin{aligned}
\int_V u(x)\left(\Delta_k^h v(x)\right) \mathrm{d}x &= \int_V u(x)\left(\frac{v(x+he_k) - v(x)}{h}\right) \mathrm{d}x \\
&= \int_{\operatorname{supp}(v) - he_i} \frac{u(x)v(x+he_k)}{h} \mathrm{d}x - \int_{\operatorname{supp}(v)} \frac{u(x)v(x)}{h} \mathrm{d}x \\
&= \int_{\operatorname{supp}(v)} \frac{u(x-he_k)v(x)}{h} \mathrm{d}x - \int_{\operatorname{supp}(v)} \frac{u(x)v(x)}{h} \mathrm{d}x \\
&= -\int_{\operatorname{supp}(v)} \frac{u(x-he_k) - u(x)}{(-h)} v(x) \, \mathrm{d}x \\
&= -\int_V \left(\Delta_k^{-h} u(x)\right) v(x) \, \mathrm{d}x.
\end{aligned}$$

Therefore, by density arguments, it follows that

$$\int_U u(x)\left(\Delta_k^h v\right) \mathrm{d}x = -\int_U \left(\Delta_k^{-h} u(x)\right) v(x) \, \mathrm{d}x$$

for every $u, v \in \mathrm{H}_0^1(U)$.

2. Observe that

$$
\begin{aligned}
\Delta_k^h(vu) &= \frac{v(x+he_k)u(x+he_k) - v(x)u(x)}{h} \\
&= \frac{v(x+he_k)u(x+he_k) - v(x+he_k)u(x) + v(x+he_k)u(x) - v(x)u(x)}{h} \\
&= \left(\tau_k^h v\right) \frac{u(x+he_k) - u(x)}{h} + u(x)\frac{v(x+he_k) - v(x)}{h} \\
&= \left(\tau_k^h v\right) \Delta_k^h u + u\Delta_k^h v.
\end{aligned}
$$

□

Lemma 5.4.5. *Suppose* $u \in L^2(U)$ *and* $V \Subset U$. *Then* $u \in \mathrm{H}^1(V)$ *if and only if*

$$\left\|\Delta^h u\right\|_{L^2(V)} \leq C$$

for all h *such that* $0 < |h| < 2\mathrm{dist}(V, \partial U)$, *and some* $C > 0$. *Moreover, there exists* $\tilde{C} > 0$ *such that*

$$\frac{1}{\tilde{C}}\|Du\|_{L^2(V)} \leq \left\|\Delta^h u\right\|_{L^2(V)} \leq \tilde{C}\|Du\|_{L^2(V)}.$$

Proof. $(\Rightarrow)$. Let $u \in \mathcal{C}^\infty(U)$. Then for $i \in \{1, \ldots, d\}$ using the fundamental theorem of calculus it follows that

$$
\begin{aligned}
|u(x+he_i) - u(x)| &= \left|\int_0^h \frac{\mathrm{d}u(x+te_i)}{\mathrm{d}t}\,\mathrm{d}t\right| \\
&\leq \int_0^h \left|\frac{\mathrm{d}u(x+te_i)}{\mathrm{d}t}\right|\,\mathrm{d}t \\
&\leq \int_0^h |Du(x+te_i)|\mathrm{d}t \\
&\overset{t=hs}{=} \int_0^1 |Du(x+she_i)||h|\,\mathrm{d}s.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_V \left|\Delta^h u(x)\right|^2\,\mathrm{d}x &= \int_V \sum_{i=1}^d \left|\Delta_i^h u(x)\right|\,\mathrm{d}x \\
&= \sum_{i=1}^d \int_V \frac{|u(x+he_i) - u(x)|}{|h|}\,\mathrm{d}x \\
&\leq \sum_{i=1}^d \int_V \int_0^1 |Du(x+she_i)|\,\mathrm{d}s\,\mathrm{d}x
\end{aligned}
$$

Therefore, applying Fubini's theorem

$$\begin{aligned}
\left\|\Delta^h u\right\|_{L^2(V)}^2 &= \int_V \left|\Delta^h u(x)\right|^2 \,\mathrm{d}x \\
&\leq \sum_{i=1}^d \int_V \int_0^1 |Du(x+she_i)|^2 \,\mathrm{d}s \,\mathrm{d}x \\
&= \sum_{i=1}^d \int_0^1 \int_V |Du(x+she_i)|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\leq \sum_{i=1}^d \int_0^1 \|Du\|_{L^2(U)}^2 \,\mathrm{d}s \\
&= d\|Du\|_{L^2(U)}^2.
\end{aligned}$$

Hence,

$$\left\|\Delta^h u\right\|_{L^2(V)} \leq \sqrt{d}\|Du\|_{L^2(U)}.$$

For $u \in \mathrm{H}^1(V)$ there exists a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}^\infty(U)$ such that $u_n \to u$ in $\mathrm{H}^1(V)$. Hence,

$$\left\|\Delta^h u\right\|_{L^2(V)} \leq \sqrt{d}\|Du\|_{L^2(U)}$$

for all $u \in \mathrm{H}^1(V)$.

$(\Leftarrow)$. Let $(h_n)_{n\in\mathbb{N}}$ be such that $0 < 2|h_n| < \operatorname{dist}(V, \partial U)$ and $h_n \to 0$ as $n \to \infty$. Note that the sequence $\left(\Delta_i^{-h_n} u\right)_{n\in\mathbb{N}}$ is bounded by C in $L^2(V)$. Therefore, as $L^2(V)$ is a Hilbert space it follows from Theorem 5.3.11 that there exists a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ such that $\left(\Delta_i^{-h_k} u\right)_{k\in\mathbb{N}} \subseteq L^2(V)$ converges weakly to some $v_i \in L^2(V)$ where $\|v_i\|_{L^2(V)} \leq C$. For $\phi \in \mathcal{C}_c^\infty(U)$ we have that $u(x)\Delta_i^h\phi(x)$ is integrable on V since $u \in L^2(V)$ and ϕ is continuous with compact support. Therefore, by the dominated convergence theorem we have that

$$\begin{aligned}
\lim_{h\to 0} \int_V u(x) \left(\Delta_i^h \phi(x)\right) \,\mathrm{d}x &= \int_V u(x) \lim_{h\to 0} \left(\Delta_i^h \phi(x)\right) \,\mathrm{d}x \\
&= \int_V u(x)\phi'(x) \,\mathrm{d}x.
\end{aligned}$$

From statement 1 of Lemma 5.4.4 we know that

$$\lim_{h\to 0} \int_V \left(\Delta_i^{-h} u(x)\right) \phi(x) \,\mathrm{d}x = \int_V v_i \phi(x) \,\mathrm{d}x.$$

Hence,

$$\begin{aligned}
\int_V u(x)\phi'(x) \,\mathrm{d}x &= \lim_{h\to 0} \int_V u(x) \left(\Delta_i^h \phi(x)\right) \,\mathrm{d}x \\
&= -\lim_{h\to 0} \int_V \left(\Delta_i^{-h} u(x)\right) \phi(x) \,\mathrm{d}x \\
&= -\int_v v_i \phi(x) \,\mathrm{d}x.
\end{aligned}$$

Therefore, $D_i u = v_i$ in the weak sense. Thus, as $\|v_i\|_{L^2(V)} \leq C$ it follows that $u \in \mathrm{H}^1(V)$. □

Remark 5.4.6. *From Lemma 5.4.5 we see that Δ^h is equivalent, in terms of the norm, to the weak derivative on compact subsets of U. Hence, establishing results for the difference quotients allows us to infer results about u and $\mathrm{H}^2_{loc}(U)$.*

Theorem 5.4.7. *Let $U \subseteq \mathbb{R}^d$ be a bounded and open set. Let $u \in \mathrm{H}^1(U)$ be a weak solution to*

$$Lu = f$$

on U where

$$Lu = -\sum_{i,j=1}^{d} \left(a_{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{d} b_i(x)u_{x_i} + c(x)u$$

is a uniformly elliptic operator with $a_{ij} \in \mathcal{C}^1(U)$, $b_i, c \in L^\infty(U)$ for each $i,j = 1,\dots,d$ and $f \in L^2(U)$. Then, $u \in \mathrm{H}^2_{loc}(U)$ and for each $V \Subset U$ we have

$$\|u\|_{\mathrm{H}^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

for $C = C\left(V, U, a_{ij}, b_i, c\right)$.

Proof. Step 1: Reformulate the problem.
Fix $V \Subset U$ and let W be such that $V \Subset W \Subset U$. Let $\xi \in \mathcal{C}_c^\infty(W)$, with $0 \leq \xi \leq 1$, $\xi \equiv 1$ on V and $\xi \equiv 0$ on ∂W. As u is a weak solution, $B[u,v] = (f,v)$ for all $v \in \mathrm{H}^1_0(U)$. In particular,

$$\sum_{i,j=1}^{d} a_{ij}u_{x_i}v_{x_j}\,\mathrm{d}x = \int_U \tilde{f}v\,\mathrm{d}x \tag{5.4.2}$$

for all $v \in \mathrm{H}^1_0(U)$ where

$$\tilde{f} := f - \sum_{i=1}^{d} b_i u_{x_i} - cu.$$

For some $k = 1,\dots,d$ take $v = -\Delta_k^{-h}\left(\xi^2\Delta_k^h u\right)$, with $A := \sum_{i,j=1}^{d}\int_U a_{ij}u_{x_i}v_{x_j}\,\mathrm{d}x$ and $B := \int_U \tilde{f}v\,\mathrm{d}x$.
Step 2: Bound A.
Observe that

$$\begin{aligned}
A &= -\sum_{i,j=1}^{d}\int_U a_{ij}u_{x_i}\Delta_k^{-h}\left(\xi^2\Delta_k^h u\right)_{x_i}\,\mathrm{d}x\\
&\overset{(1)}{=} \sum_{i,j=1}^{d}\int_U \Delta_k^h\left(a_{ij}u_{x_i}\right)\left(\xi^2\Delta_k^h u\right)_{x_j}\,\mathrm{d}x\\
&\overset{(2)}{=} \sum_{i,j=1}^{d}\int_U \left(\tau_k^h a_{ij}\Delta_k^h u_{x_i} + \left(\Delta_k^h a_{ij}\right)u_{x_i}\right)\left(\xi^2\Delta_k^h u_{x_j} + 2\xi\xi_{x_j}\Delta_k^h u\right)\,\mathrm{d}x
\end{aligned}$$

where (1) is an application of statement 1 of Lemma 5.4.4, and for (2) we have used statement 2 of Lemma 5.4.4 and the product rule for weak derivatives which can be applied as ξ is smooth. In particular, one can write $A = A_1 + A_2$ where

$$A_1 = \sum_{i,j=1}^{d}\int_U \xi^2\left(\tau_k^h a_{ij}\right)\left(\Delta_k^h u_{x_i}\right)\left(\Delta_k^h u_{x_j}\right)\,\mathrm{d}x$$

and

$$A_2 = \sum_{i,j=1}^{d}\int_U \underbrace{\left(\Delta_k^h a_{ij}\right)u_{x_i}\xi^2\Delta_k^h u_{x_j}}_{\leq C_1\xi\left|\Delta_k^h(Du)\right||Du|} + \underbrace{\left(\Delta_k^h a_{ij}\right)u_{x_i}\left(2\xi\xi_{x_j}\Delta_k^h u\right)}_{\leq C_2\xi|Du|\left|\Delta_k^h u\right|} + \underbrace{\left(\tau_k^h a_{ij}\right)\left(\Delta_k^h u_{x_i}\right)\left(2\xi\xi_{x_j}\Delta_k^h u\right)}_{\leq C_3\xi\left|\Delta_k^h(Du)\right|\left|\Delta_k^h u\right|}\,\mathrm{d}x,$$

where the inequalities arise as $a_{ij} \in \mathcal{C}^1(U)$ and ξ is supported on the compact set W. By uniform ellipticity we know that

$$\sum_{i,j=1}^{d}\left(\tau_k^h a_{ij}\right)\zeta_i\zeta_j \geq \theta|\zeta|^2$$

for all $\zeta \in \mathbb{R}^d$. Letting $\zeta_i = \Delta_k^h u_{x_i} = \left(\Delta_k^h u\right)_{x_i}$ it follows that

$$A_1 \geq \theta \int_U \xi^2 \left|\Delta_k^h(Du)\right|^2 \,\mathrm{d}x. \tag{5.4.3}$$

As

$$|A_2| \leq C \int_W \xi \left|\Delta_k^h(Du)\right| |Du| + \xi |Du| \left|\Delta_k^h u\right| + \xi \left|\Delta_k^h(Du)\right| \left|\Delta_k^h u\right| \,\mathrm{d}x,$$

we can apply Lemma 4.6.1 to deduce that

$$|A_2| \leq \epsilon \int_W \xi^2 \left|\Delta_k^h(Du)\right|^2 \,\mathrm{d}x + \frac{C}{\epsilon} \int_W |Du|^2 + \left|\Delta_k^h u\right|^2 \,\mathrm{d}x \tag{5.4.4}$$

for all $\epsilon > 0$. From Lemma 5.4.5 we have that

$$\int_W \left|\Delta_k^h u\right|^2 \,\mathrm{d}x \leq \tilde{C} \int_W |Du|^2 \,\mathrm{d}x,$$

thus setting $\epsilon = \frac{\theta}{2}$ in (5.4.4) and using (5.4.3) it follows that

$$\begin{aligned}
A &= A_1 + A_2 \\
&\geq \theta \int_W \xi^2 \left|\Delta_k^h(Du)\right|^2 \,\mathrm{d}x - |A_2| \\
&\geq \frac{\theta}{2} \int_W \xi^2 \left|\Delta_k^h(Du)\right|^2 \,\mathrm{d}x - C \int_W |Du|^2 \,\mathrm{d}x
\end{aligned}$$

for some $C > 0$.

Step 3: Bound B.

Observe that

$$|B| \leq C \int_U (|f| + |Du| + |u|) \, \Delta_k^{-h} \left(\xi^2 \Delta_k^h u\right) \,\mathrm{d}x \tag{5.4.5}$$

for some constant $C = C(b, c)$. Using Lemma 5.4.5 it follows that

$$\begin{aligned}
\int_U \left|\Delta_k^{-h} \left(\xi^2 \Delta_k^h u\right)\right|^2 \,\mathrm{d}x &\leq C \int_U \left|D\left(\xi^2 \Delta_k^h u\right)\right|^2 \,\mathrm{d}x \\
&\leq C \int 2\xi \xi_{x_i} \left|\Delta_k^h u\right|^2 + \xi^2 \left|\Delta_k^h(Du)\right|^2 \,\mathrm{d}x \\
&\leq C \int_U |Du|^2 \,\mathrm{d}x + C \int_U \xi^2 \left|\Delta_k^h(Du)\right|^2 \,\mathrm{d}x.
\end{aligned}$$

where we have used $(a+b)^2 \leq 2a^2 + 2b^2$. Thus, applying Lemma 4.6.1 to (5.4.5) we deduce that

$$|B| \leq \epsilon \int_U \xi^2 \left|\Delta_k^h(Du)\right|^2 \,\mathrm{d}x + C \int_W |f|^2 + |u|^2 + |Du|^2 \,\mathrm{d}x \tag{5.4.6}$$

for all $\epsilon > 0$.

Step 4: Form bound.

Recall from (5.4.2) that $A + B = 0$ which implies that $|A| = |B|$. Using this, step 2 and setting $\epsilon = \frac{\theta}{4}$ in (5.4.6) we deduce that

$$\int_U \xi^2 \left|\Delta_k^h(Du)\right| \,\mathrm{d}x \leq C \int_W |f|^2 + |u|^2 + |Du|^2 \,\mathrm{d}x.$$

Since $\xi \equiv 1$ on $V \Subset U$ we get that

$$\int_V \left|\Delta_k^h(Du)\right|^2 \,\mathrm{d}x \leq C \int_W |f|^2 + |u|^2 + |Du|^2 \,\mathrm{d}x < \infty$$

for $u \in \mathrm{H}^1(V)$ and with C independent of h. Therefore, by Lemma 5.4.5 we have that $Du \in \mathrm{H}^1(U)$ so that $u \in \mathrm{H}^1_{\text{loc}}(U)$. In particular, let $h \searrow 0$ to deduce that

$$\left\|D^2 u\right\|_{L^2(V)} \leq C \int_W |f|^2 + |u|^2 + |Du|^2 \,\mathrm{d}x. \tag{5.4.7}$$

<u>Step 5: Refine bound.</u>
Let $\xi \in \mathcal{C}_c^\infty(U)$ with $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on W. Set $v = \xi^2 u$ in the we weak formulation $B[u,v] = (f,v)$ to deduce that

$$\int_U \sum_{i,j=1}^d a_{ij} u_{x_i} \left(\xi^2 u\right)_{x_j} + \sum_{i=1}^d b_i u_{x_i} \xi^2 u + c\xi^2 u^2 \,\mathrm{d}x = \int_U f\xi^2 u \,\mathrm{d}x.$$

Using similar steps as those made to show (5.3.3) it follows that

$$\|Du\|_{L^2(W)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

for some $C > 0$. Therefore, using (5.4.7) and noting that $\|\cdot\|_{L^2(W)} \leq \|\cdot\|_{L^2(U)}$ we deduce that

$$\|u\|_{\mathrm{H}^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

for some constant $C > 0$. □

Remark 5.4.8.

1. *Note we are not requiring $u = 0$ on ∂U, in the trace sense, namely we only require $u \in \mathrm{H}^1(U)$ rather than $\mathrm{H}^1_0(U)$.*
2. *Let $v \in \mathcal{C}^\infty(U)$, then $B[u,v] = (f,v)$. Since, $u \in \mathrm{H}^2_{loc}(U)$ we can integrate by parts to get that*

   $$B[u,v] = (Lu, v).$$

   Thus, $(Lu - f, v) = 0$ for all $v \in \mathcal{C}_c^\infty(U)$. Consequently, $Lu = f$ almost everywhere in U.
3. *Theorem 5.4.7 is a local result. Namely, it says that if f behaves well in a region then so will u. That is, singularities do not propagate in from the boundary or from regions where f is not well behaved.*
4. *As we only require uniform ellipticity on compact subsets, there can be degeneracy near the boundary ∂U.*

Exercise 5.4.9. *Let $U \subseteq \mathbb{R}^d$ for $d \geq 3$ and ∂U a $\mathcal{C}^2$-boundary. Consider*

$$\begin{cases} -\Delta u + u + |u|^p = f & U \\ u = 0 & \partial U \end{cases}$$

for $p > 1$. Show that if $f \in L^2(U)$ so that $\|f\|_{L^2(U)} < \epsilon$, then there exists a solution $u \in \mathrm{H}^2(U)$.

Theorem 5.4.10. *Let $U \subseteq \mathbb{R}^d$ be a bounded and open set. Let $u \in \mathrm{H}^1(U)$ be a weak solution to*

$$Lu = f$$

on U where

$$Lu = -\sum_{i,j=1}^d \left(a_{ij}(x) u_{x_i}\right)_{x_j} + \sum_{i=1}^d b_i(x) u_{x_i} + c(x)u$$

is a uniformly elliptic operator with $a_{ij}, b_i, c \in \mathcal{C}^{m+1}(U)$ for each $i,j = 1,\ldots,d$ and $f \in \mathrm{H}^m(U)$. Then,

$u \in \mathrm{H}^{m+2}_{loc}(U)$ and for each $V \Subset U$ we have

$$\|u\|_{\mathrm{H}^{m+2}(V)} \leq C\left(\|f\|_{\mathrm{H}^m(U)} + \|u\|_{L^2(U)}\right) \tag{5.4.8}$$

for $C = C\left(V, U, a_{ij}, b_i, c\right)$.

Proof.

- The case $m = 0$ is the statement of Theorem 5.4.7.
- Assume that for uniformly elliptic operators L with $a_{ij}, b_i, c \in \mathcal{C}^{m+1}(U)$ and $f \in \mathrm{H}^m(U)$, weak solutions u are $\mathrm{H}^{m+2}_{\mathrm{loc}}(U)$ regular and such that for any $V \Subset W \Subset U$ (5.4.8) holds. Now suppose that L is a uniformly elliptic operator with $a_{ij}, b_i, c \in \mathcal{C}^{m+2}(U)$, $f \in \mathrm{H}^{m+1}(U)$ and fix $V \Subset W \Subset U$. In particular, note that we can apply the inductive assumption to deduce that
$$\|u\|_{\mathrm{H}^{m+2}(V)} \leq C\left(\|f\|_{\mathrm{H}^m(U)} + \|u\|_{L^2(U)}\right). \tag{5.4.9}$$
Consider $|\alpha| = m+1$ and let $\tilde{v} \in \mathcal{C}^\infty_c(W)$. For a weak solution u we have that $B[u,v] = (f,v)$ for any v. In particular, let $v := (-1)^{|\alpha|} D^\alpha \tilde{v}$ and perform integration by parts to deduce that
$$B\left[D^\alpha u, \tilde{v}\right] = \left(\tilde{f}, \tilde{v}\right)$$
where
$$\tilde{f} = D^\alpha f - \sum_{\beta \leq \alpha, \beta \neq \alpha} \left(-\sum_{i,j=1}^{d} \left(D^{\alpha-\beta} a_{ij} D^\beta u_{x_i}\right)_{x_j} + \sum_{i=1}^{d} D^{\alpha-\beta} b_i D^\beta u_{x_i} + D^{\alpha-\beta} c D^\beta u\right) \tag{5.4.10}$$
and $B[\cdot,\cdot]$ is the bilinear operator corresponding to a uniformly elliptic operator L satisfying the conditions of Theorem 5.4.7. Moreover, note that $\tilde{u} := D^\alpha u \in \mathrm{H}^1(W)$. Hence, $\tilde{u}$ is a weak solution to $L\tilde{u} = \tilde{f}$ in W. From (5.4.10), our assumptions and (5.4.9) it follows that $\tilde{f} \in L^2(W)$ with
$$\left\|\tilde{f}\right\|_{L^2(W)} \leq C\left(\|f\|_{\mathrm{H}^{m+1}(U)} + \|u\|_{L^2(U)}\right). \tag{5.4.11}$$
So using Theorem 5.4.7 we get that $\tilde{u} \in \mathrm{H}^2(V)$ along with the estimate
$$\begin{aligned} \|\tilde{u}\|_{\mathrm{H}^2(V)} &\leq C\left(\left\|\tilde{f}\right\|_{L^2(W)} + \|\tilde{u}\|_{L^2(W)}\right) \\ &\overset{(5.4.11)}{\leq} C\left(\|f\|_{\mathrm{H}^{m+1}(U)} + \|u\|_{L^2(U)}\right), \end{aligned}$$
which holds for all $\alpha \in \mathbb{N}^d$ with $|\alpha| = m+1$. Thus, as $\tilde{u} = D^\alpha u$ it follows that $u \in \mathrm{H}^{m+3}(V)$ with
$$\|u\|_{\mathrm{H}^{m+3}(V)} \leq C\left(\|f\|_{\mathrm{H}^{m+1}(U)} + \|u\|_{L^2(U)}\right).$$

We conclude by induction. ☐

Corollary 5.4.11. *Let $U \subseteq \mathbb{R}^d$ be a bounded and open set. Let $u \in \mathrm{H}^1(U)$ be a weak solution to*
$$Lu = f$$
on U where
$$Lu = -\sum_{i,j=1}^{d} \left(a_{ij}(x) u_{x_i}\right)_{x_j} + \sum_{i=1}^{d} b_i(x) u_{x_i} + c(x) u$$
is a uniformly elliptic operator with $a_{ij}, b_i, c \in \mathcal{C}^\infty(U)$ for each $i,j = 1, \dots, d$ and $f \in \mathcal{C}^\infty(U)$. Then $u \in \mathcal{C}^\infty(U)$.

Proof. Using Theorem 5.4.10 it follows that $u \in \mathrm{H}^m_{\mathrm{loc}}(U)$ for every $m \in \mathbb{N}$. Thus, from Corollary 4.7.15 we have that $u \in \mathcal{C}^k(U)$ for every $k \in \mathbb{N}$ and thus $u \in \mathcal{C}^\infty(U)$. ☐

Remark 5.4.12.

1. *There is an equivalent form of Theorem 5.4.10 for Hölder spaces. Namely, for* $0 \le \gamma < 1$ *if* $f \in \mathcal{C}^{k,\gamma}(U)$ *then* $u \in \mathcal{C}^{k+2,\gamma}(U)$.

2. *Combining Theorem 5.4.10 with Sobolev embeddings, we deduce that for* $m \in \mathbb{N}$ *large enough,* $a_{ij}, b_i, c \in \mathcal{C}^{m+1}$ *and* $f \in \mathrm{H}^m$ *we have that* $u \in \mathrm{H}^{m+2}_{loc} \hookrightarrow \mathcal{C}^2_{loc}(U)$. *That is, we eventually recover a classical solution.*

Provided ∂U is regular enough, we can extend regularity results up to the boundary.

Theorem 5.4.13. *Let* $U \subseteq \mathbb{R}^d$ *be a bounded open set with* $\mathcal{C}^2$*-boundary. Let* $u \in \mathrm{H}^1_0(U)$ *be a weak solution to*

$$\begin{cases} Lu = f & U \\ u = 0 & \partial U, \end{cases}$$

where $a_{ij} \in \mathcal{C}^1\left(\bar{U}\right)$, $b_i, c \in L^\infty(U)$ *and* $f \in L^2(U)$. *Then* $u \in \mathrm{H}^2(U)$ *and*

$$\|u\|_{\mathrm{H}^2(U)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right).$$

Moreover, if u *is the unique weak solution, then*

$$\|u\|_{\mathrm{H}^2(U)} \le C\|f\|_{L^2(U)} = C\|Lu\|_{L^2(U)}.$$

Proof (Sketch). Let us restrict to the case of a flat boundary, namely,

$$U = B_1(0) \cap \{x_d = 0\}.$$

Let $V = B_{\frac{1}{2}}(0) \cap \{x_d > 0\}$, and $\xi \in \mathcal{C}^\infty_c(B_1(0))$ where $0 \le \xi \le 1$ with $\xi = 1$ on V. For u a weak solution, we have

$$\sum_{i,j=1}^d \int_U a_{ij} u_{x_i} v_{x_j} \,\mathrm{d}x = \int_U \tilde{f} v \,\mathrm{d}x \tag{5.4.12}$$

where

$$\tilde{f} := f - \sum_{i=1}^d b_i u_{x_i} - cu$$

for all $v \in \mathrm{H}^1_0(U)$. For fixed $k \in \{1, \dots, d-1\}$ and h small enough let

$$\begin{aligned} v &:= -\Delta_k^{-h}\left(\xi^2 \Delta_k^h u\right) \\ &= -\frac{1}{h}\Delta_k^{-h}\left(\xi^2(x)\left(u(x+he_k) - u(x)\right)\right) \\ &= -\frac{1}{h^2}\left(\xi^2(x-he_k)\left(u(x) - u(x-he_k)\right) - \xi^2(x)\left(u(x+he_k) - u(x)\right)\right) \end{aligned}$$

for $x \in U$. Since $u = 0$ along $\{x_d = 0\}$, in the trace sense, and $\xi = 0$ near the boundary it follows that $v \in \mathrm{H}^1_0(U)$. Thus, we can substitute this v into (5.4.12) and deduce in the same way as in the proof of Theorem 5.4.7 that

$$\int \left|\Delta_k^h(Du)\right|^2 \,\mathrm{d}x \le C\int_U \left(|f|^2 + |u|^2 + |Du|^2\right) \,\mathrm{d}x. \tag{5.4.13}$$

Thus, we can control $D_i D_k u$ for all $i, k \in \{1, \dots, d-1\}$, that is the directions tangent to the boundary. From statement 2 of Remark 5.4.8 we recall that $Lu = f$ holds pointwise almost everywhere in U. Hence,

$$-\sum_{i,j=1}^d \left(a_{ij} u_{x_i}\right)_{x_j} + \sum_{i=1}^d b_i u_{x_i} + cu = f$$

almost everywhere in U. This implies that

$$a_{dd}u_{x_dx_d} = -\sum_{i,j,i+j<2d} a_{ij}u_{x_ix_j} + \sum_{i=1}^{d} \tilde{b}_i u_{x_i} + cu - f =: F,$$

where we note $F \in L^2(U)$. Using the uniform ellipticity of L we know that $a_{dd} > 0$ almost everywhere which implies that $u_{x_dx_d} \in L^2(U)$. From (5.4.13) we have

$$\|D_k D_i u\|_{L^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{\mathrm{H}^1(U)}\right)$$

which implies that

$$\|F\|_{L^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{\mathrm{H}^1(U)}\right)$$

and so $\|u_{x_dx_d}\|_{L^2(V)}$ can be bounded by a similar expression. Therefore,

$$\|u\|_{\mathrm{H}^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{\mathrm{H}^1(U)}\right).$$

Using similar arguments as those made in the proof of (5.3.3), we can replace the $\|u\|_{\mathrm{H}^1(U)}$ on the right-hand side with $\|u\|_{L^2(U)}$ to conclude that

$$\|u\|_{\mathrm{H}^2(U)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right).$$

If u is additionally the unique weak solution, then we can apply Theorem 5.3.38 to deduce that

$$\|u\|_{\mathrm{H}^2(U)} \leq C\|f\|_{L^2(U)} = C\|Lu\|_{L^2(U)}.$$

□

5.5 Solution to Exercises

Exercise 5.3.20

Solution. Let $(u_n)_{n\in\mathbb{N}} \subseteq H$ be a bounded sequence. Then since T is a bounded linear operator, the sequence $(T(u_n))_{n\in\mathbb{N}} \subseteq H$ is also bounded. Thus, as K is compact, the sequence, $(K(T(u_n)))_{n\in\mathbb{N}} = ((K\circ T)(u_n))_{n\in\mathbb{N}}$ has a strongly convergent subsequence in H. Therefore, $K \circ T$ is a compact linear operator. □

Exercise 5.3.24

Solution. Through repeated applications of integration by parts, it follows that

$$\begin{aligned}
(L\varphi, \psi)_{L^2(U)} &= \int_U \left(-\sum_{i,j=1}^{d} \left(a_{ij}(x)\varphi_{x_i}\right)_{x_j} + \sum_{i=1}^{d} b_i(x)\varphi_{x_i} + c(x)\varphi \right) \psi \,\mathrm{d}x \\
&= \int_U \sum_{i,j=1}^{d} a_{ij}(x)\varphi_{x_j}\psi_{x_i} - \sum_{i=1}^{d} \left(b_i(x)\psi\right)_{x_i} \varphi + c(x)\varphi\psi \,\mathrm{d}x \\
&= \int_U -\sum_{i,j=1}^{d} \left(a_{ij}(x)\psi_{x_i}\right)_{x_j} \varphi - \sum_{i=1}^{d} \left(b_i(x)\psi_{x_i} + \left(b_i(x)\right)_{x_i} \psi\right) \varphi + c(x)\psi\varphi \,\mathrm{d}x \\
&= \int_U \left(L^*\psi\right) \varphi \,\mathrm{d}x \\
&= \left(\varphi, L^*\psi\right)_{L^2(U)}.
\end{aligned}$$

□

Exercise 5.3.27

Solution. Take $\phi \in \mathcal{C}_c^\infty(U)$, then $a_{ij}\phi_{x_j}$ is compactly supported and in $L^2(U)$. Consequently, for $i = 1, \ldots, d$ we can convolve $a_{ij}\phi_{x_j}$ with a mollifier to construct a sequence $\left(\psi_m^i\right)_{m\in\mathbb{N}} \subseteq \mathcal{C}_c^\infty(U)$ such that

$$\left\| \psi_m^i - \sum_{j=1}^d a_{ij}\phi_{x_j} \right\|_{L^2(U)} < \frac{1}{2^m}.$$

Let

$$f_m := -\sum_{i=1}^d \left(\psi_m^i\right)_{x_i} + \sum_{i=1}^d b_i(x)\phi_{x_i} + (c(x)+\gamma)\phi,$$

which we note is in $L^2(U)$ and has compact support. Let $\phi_m := L_\gamma^{-1}(f_m)$. Consider $u \in \mathrm{H}_0^1(U)$ and let

$$B_\gamma[\cdot,\cdot] = B[\cdot,\cdot] + \gamma(\cdot,\cdot)_{L^2(U)}$$

where B is the bilinear operator associated to L. Then

$$\begin{aligned} B_\gamma[\phi_m, u] &= (f_m, u)_{L^2(U)} \\ &= \int_U \sum_{i=1}^d \psi_m^i u_{x_i} + \sum_{i=1}^d b_i(x)\phi_{x_i} u + (c(x)+\gamma)\phi u \,\mathrm{d}x. \end{aligned}$$

Hence,

$$\begin{aligned} B_\gamma\left[\phi_m - \phi, \phi_m - \phi\right] &= B_\gamma\left[\phi_m, \phi_m\right] - B_\gamma[\phi, \phi_m] - B_\gamma[\phi_m, \phi] + B_\gamma[\phi, \phi] \\ &= \int_U \sum_{i=1}^d \psi_m^i \left(\phi_m\right)_{x_i} + \sum_{i=1}^d b_i(x)\phi_{x_i}\phi_m + (c(x)+\gamma)\phi\phi_m \,\mathrm{d}x \\ &\quad - \int_U \sum_{i=1}^d \psi_m^i \phi_{x_i} + \sum_{i=1}^d b_i(x)\phi_{x_i}\phi + (c(x)+\gamma)\phi\phi \,\mathrm{d}x \\ &\quad - \int_U \sum_{i=1}^d a_{ij}\phi_{x_i}\left(\phi_m\right)_{x_j} + \sum_{i=1}^d b_i(x)\phi_{x_i}\phi_m + (c(x)+\gamma)\phi\phi_m \,\mathrm{d}x \\ &\quad + \int_U \sum_{i=1}^d a_{ij}\phi_{x_i}\phi_{x_j} + \sum_{i=1}^d b_i(x)\phi_{x_i}\phi + (c(x)+\gamma)\phi\phi \,\mathrm{d}x \\ &= \int_U \sum_{i=1}^d \left(\psi_m^i - \sum_{j=1}^d a_{ij}\phi_{x_j}\left(\phi_m - \phi\right)_{x_i} \right) \mathrm{d}x \\ &\le d \sup_{i=1,\ldots,d} \left\| \psi_m^i - \sum_{j=1}^d a_{ij}\phi_{x_j} \right\|_{L^2(U)} \|\phi_m - \phi\|_{\mathrm{H}^1(U)} \\ &\le \frac{d}{2^m} \left\|\phi_m - \phi\right\|_{\mathrm{H}^1(U)}. \end{aligned} \tag{5.5.1}$$

Recall from (5.3.3) that

$$B_\gamma[v,v] \ge \beta\|v\|_{\mathrm{H}^1(U)}^2$$

for some $\beta > 0$ and all $v \in \mathrm{H}_0^1(U)$. Hence, using (5.5.1) we get that

$$\beta\left\|\phi_m - \phi\right\|_{\mathrm{H}^1(U)}^2 \le \frac{d}{2^m}\left\|\phi_m - \phi\right\|_{\mathrm{H}^1(U)},$$

thus,

$$\left\|\phi_m - \phi\right\|_{\mathrm{H}^1(U)} \le \frac{d}{2^m\beta} \overset{m\to\infty}{\longrightarrow} 0.$$

Therefore, $\phi_m \to \phi$ in $\mathrm{H}^1(U)$ and so we conclude by recalling that $\phi \in \mathcal{C}_c^\infty(U)$ is dense in $\mathrm{H}_0^1(U)$. $\square$

Exercise 5.4.9

Solution. Let $\Gamma : \mathrm{H}^2(U) \to \mathrm{H}^1_0(U)$ take $w \in \mathrm{H}^2(U)$ to a solution $u \in \mathrm{H}^1_0(U)$ of

$$\begin{cases} -\Delta u + u = f - |w|^p & U \\ u = 0 & \partial U. \end{cases}$$

Consider $B_b := \left\{u \in \mathrm{H}^2(U) : \|u\|_{\mathrm{H}^2(U)} \le b\right\}$ for some $b > 0$. Note that $\left(B_b, \|\cdot\|_{\mathrm{H}^2(U)}\right)$ is a complete metric space as it is a closed ball in a Banach space.

Step 1: The map $\Gamma : B_b \to B_b$ is well-defined.

Let $w \in B_b$. Then by Theorem 4.7.13 it follows that $w \in \mathcal{C}^{0,\frac{1}{2}}\left(\bar{U}\right)$ and there exists a constant $C_1 = C_1(U)$ such that

$$\|w\|_{\mathcal{C}^{0,\frac{1}{2}}\left(\bar{U}\right)} \le C_1 \|w\|_{\mathrm{H}^2(U)} \le C_1 b. \tag{5.5.2}$$

By Hölder's inequality it follows that

$$\begin{aligned} \||w|^p\|_{L^2(U)} &\le \left(\sup_U |w|^p\right)\left(\int_U |w|^p\right)^{\frac{1}{2}} \\ &\le \||w|^p\|^{\frac{1}{2}}_{L^\infty(U)} \||w|^p\|^{\frac{1}{2}}_{L^\infty(U)} |U|^{\frac{1}{2}} \\ &= \||w|^p\|_{L^\infty(U)} |U|^{\frac{1}{2}} \\ &\le |U|^{\frac{1}{2}} (C_1 b)^p \\ &=: b^p C_2. \end{aligned}$$

Moreover, by Theorem 5.4.7, given $g \in L^2(U)$ the linear elliptic problem

$$\begin{cases} -\Delta u + u = g & U \\ u = 0 & \partial U \end{cases}$$

admits a unique solution $\mathrm{H}^1_0(U) \cap \mathrm{H}^2(U)$ and there exists a constant C_3 such that

$$\|u\|_{\mathrm{H}^2(U)} \le C_3 \|g\|_{L^2(U)}.$$

Applying this to $g = f - |w|^p$ it follows that

$$\begin{aligned} \|u\|_{\mathrm{H}^2(U)} &\le C_3 \left\|f - |w|^p\right\|_{L^2(U)} \\ &\le C_3 \left(\epsilon + C_2 b^p\right). \end{aligned}$$

So letting $b^{p-1} < \left(2C_2C_3\right)^{-1}$ and $\epsilon < b\left(2C_3\right)^{-1}$ it follows that $\|\Gamma(w)\|_{\mathrm{H}^2(U)} \le b$.

Step 2: The map $\Gamma : B_b \to B_b$ is a contraction.

Let $u_i = \Gamma(w_i)$ for $i = 1, 2$. By linearity it follows that $u = u_1 - u_2 \in \mathrm{H}^1_0(U) \cap \mathrm{H}^1(U)$ is the unique solution of

$$\begin{cases} -\Delta u + u = |w_2|^p - |w_1|^p & U \\ u = 0 & \partial U. \end{cases}$$

Using Theorem 5.4.7 we have

$$\begin{aligned} \|u_1 - u_2\|_{\mathrm{H}^2(U)} &= \|\Gamma(w_1) - \Gamma(w_2)\|_{\mathrm{H}^2(U)} \\ &\le C_3 \left\||w_1|^p - |w_2|^p\right\|_{L^2(U)}. \end{aligned} \tag{5.5.3}$$

Using

$$\begin{aligned} |a^p - b^p| &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(t^p\right) \mathrm{d}t \\ &= \int_a^b p t^{p-1} \,\mathrm{d}t \\ &\le |a - b| \max\left(a^{p-1}, b^{p-1}\right) \\ &\le |a - b| \left(a^{p-1} + b^{p-1}\right) \end{aligned}$$

with $a = |w_1(x)|^p$ and $b = |w_2(x)|^p$, it follows that

$$\begin{aligned}||w_1(x)|^p - |w_2(x)|^p| &\leq ||w_1(x)| - |w_2(x)|| \left(|w_1(x)|^{p-1} + |w_2(x)|^{p-1}\right) \\ &\leq ||w_1(x)| - |w_2(x)|| \left(\|w_1\|^{p-1}_{L^\infty(U)} + \|w_2\|^{p-1}_{L^\infty(U)}\right) \\ &\leq |w_1(x) - w_2(x)| \left(\|w_1\|^{p-1}_{L^\infty(U)} + \|w_2\|^{p-1}_{L^\infty(U)}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\||w_1|^p - |w_2|^p\|_{L^2(U)} &\leq \|w_1 - w_2\|_{L^2(U)} \left(\|w_1(x)\|^{p-1}_{L^\infty(U)} + \|w_2\|^{p-1}_{L^\infty(U)}\right) \\ &\overset{(1)}{\leq} \|w_1 - w_2\|_{L^2(U)} \left(2(C_1 b)^{p-1}\right)\end{aligned}$$

where in (1) we have used (5.5.2) to note that

$$\|w\|_{L^\infty(\bar{U})} \leq \|w\|_{\mathcal{C}^{0,\frac{1}{2}}(\bar{U})} \leq C_1 b.$$

Returning to (5.5.3) it follows that

$$\|\Gamma(w_1) - \Gamma(w_2)\|_{\mathrm{H}^2(U)} \leq \left(C_3 2 \left(C_1 b\right)^{p-1}\right) \|w_1 - w_2\|_{L^2(U)}.$$

Provided b is sufficiently small, we have $\left(C_3 2 \left(C_1 b\right)^{p-1}\right) < 1$ and thus $\Gamma : B_b \to B_b$ is a contraction.

Step 3: Apply Banach's fixed point theorem.

As B_b is a Banach space and Γ is a contraction, it follows that Γ has a unique fixed point in B_b. That is, there exists a $u \in B_b$ such that

$$\begin{cases} -\Delta u + u = f - |u|^p & U \\ u = 0 & \partial U \end{cases}$$

and thus u is a solution to our problem. □

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