Dynamical Systems

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1 Motivating Example

Here we are interested in the analysis of dynamical systems that evolve according to maps on a state space. As a recurring example consider the mapping $E_k : [0, 1) \to [0, 1)$ defined by

$$E_k(x) = kx \bmod 1.$$

One can think about this map in different ways.

1. E_k is a map of the unit circle $S^1 \simeq \mathbb{R}/\mathbb{Z} \simeq [0, 1)$, which uniformly stretches the circle and wraps it back around the circle with each application.

2. In base k, x can be represented as $0.x_0x_1x_2\ldots$ for $x_i \in \{0, \ldots, k-1\}$. Then E_k has the effect of removing the digit immediately to the right of the decimal point,

$$E_k(0.x_0x_1x_2\ldots) = 0.x_1x_2\ldots.$$

We proceed by considering E_k from the second perspective. Note that the base k approximation $x \approx 0.x_0\ldots x_n$ has a precision of $k^{-(n+1)}$. Therefore, in the limit, we can exactly represent any value in $[0, 1)$. More specifically, we have that $x = 0.x_0\ldots x_{n-1}$ if and only if $x \in \left[\sum_{i=0}^{n-1} x_i k^{-(i+1)}, k^{-n} + \sum_{i=0}^{n-1} x_i k^{-(i+1)}\right]$. Consequently, we can make the following deductions.

1. We can find $x \in [0, 1)$ such the orbit of x under E_k, $O^+_{E_k}(x) = \{x, E_k(x), E_k^2(x), \ldots\}$, intersects every open subset of $[0, 1)$.

2. For every opens subset of $[0, 1)$ we can find an x in this subset, such that its orbit is periodic.

3. For a fixed n, and for any $\tilde{x} \in [0, 1)$, we can find an $x \in \left[\sum_{i=0}^{n-1} x_i k^{-(n+1)}, k^{-n} + \sum_{i=0}^{n-1} x_i k^{-(n+1)}\right]$ such that $E_k(x) = \tilde{x}$.

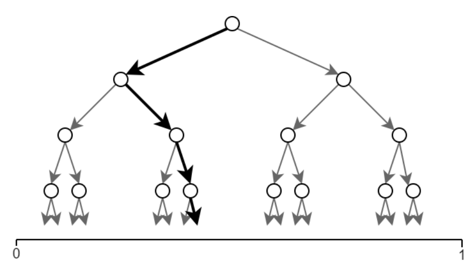


Figure 1:

We will denote the space of these semi-infinite base k expansions as Σ_k^+. Our perspective that E_k is a shift map, allows us to describe its dynamics with

$$\sigma(x_0x_1x_2\ldots) = x_1x_2\ldots.$$

In particular, with $h : \Sigma_k^+ \to [0, 1)$ defined as

$$h(x_0x_1\ldots) = \sum_{i=0}^{\infty} x_i k^{-(i+1)}$$

we can transition between these representations by noting that

$$E_k \circ h = h \circ \sigma.$$

Our aim will be to study the conceptually simpler shift dynamical system given by $\sigma : \Sigma_k^k \to \Sigma_k^+$. To do this it will be useful to introduce a metric $d^{\Sigma^+} : \Sigma_k^+ \times \Sigma_k^+ \to \mathbb{R}$ which is defined as

$$d^{\Sigma^+}(x, y) = \sum_{i=0}^{\infty} \frac{\delta(x_i, y_i)}{3^i}.$$

We would like to be able to answer questions on how often orbits visit subsets of $[0, 1)$. In particular, we like to work with

$$F(A)(n, x) := \frac{1}{n} \sum_{i=0}^{n-1} \chi_A \left(E_k^i(x) \right)$$

where χ_A is the indicator functor for $A \subset [0, 1)$. There are some nuances we will have to consider when approaching this question, as one can relatively easily construct expansions that have extraordinary behaviour for certain subsets of $[0, 1)$.

2 Topological Dynamics

2.1 Continuous Maps and Their Orbits

We consider dynamical systems in discrete time, on a state space X, propagated by the continuous map $f : X \to X$.

- We assume X is a compact metric space with the metric $d^X : X \times X \to \mathbb{R}$.
- An element $x \in X$ represents a state.

We can evolve the system n-steps forward in time with $f^n = \underbrace{f \circ \cdots \circ f}_{n}$. The forward orbit of a state is $O_f^+(x) := \{x, f(x), f^2(x), \dots\}$.

- A point $x \in X$ is a fixed point if $O_f^+ = \{x\}$.
- An orbit is periodic if $O_f^+ = \{x, f(x), \dots, f^{p-1}(x)\}$.
  - The least such p for which this holds is called the period.

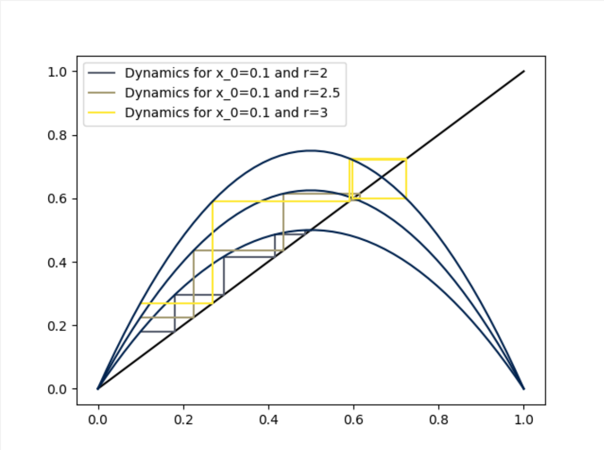
Theorem 2.1.1. *Let $f : I \to I$ be a continuous map of the interval with a periodic orbit of period 3. Then f has periodic orbits of any period.*

The Sharkovskii ordering of natural numbers is defined as

$$3 \triangleright 5 \triangleright 7 \triangleright \cdots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \cdots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright \cdots \triangleright 2^m \cdot 3 \triangleright 2^m \cdot 5 \triangleright 2^m \cdot 7 \triangleright \cdots \triangleright 2^3 \triangleright 2^2 \triangleright 1.$$

Theorem 2.1.2 (Sharkovskii)**.** *Let $I \subset \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ be continuous. If f has a periodic orbit of period n, then f has m-periodic points for all $n \triangleright m$.*

Example 2.1.3. *The logistic map $f_r(x) = rx(1-x)$ for $0 < r \le 4$ gives rise to interesting dynamics as we vary the value of r and change the initial conditions.*



2.2 ω-limit Sets, Invariant Sets and Attractors

Often we like to determine the long-term dynamics of a system. Even if the short-term dynamics of a system may seem complex and do not conform to any pattern, more often than not, the long-term dynamics settle into identifiable behaviour. The obvious way to proceed would be to investigate the quantity, $\lim_{n\to\infty} f^n(x)$.

- If this exists then the point x is necessarily a fixed point of f.
- However, if x admits a periodic orbit then this would not exist.

Instead, we can defer to subsequences to identify local features of the dynamics.

Definition 2.2.1. *A point $\tilde{x} \in X$ is an ω-limit point of $x \in X$ for a continuous map $f : X \to X$ if there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $\lim_{k\to\infty} f^{n_k}(x) = \tilde{x}$.*

Note that being strictly increasing means that $\lim_{k\to\infty} n_k = \infty$, and so we are indeed capturing long-term patterns with an ω-limit point.

Definition 2.2.2. *The ω-limit set of a point $x \in X$, denoted $\omega(x)$, for a continuous map $f : X \to X$ is the set of all ω-limit points of x.*

As we are dealing with compact state spaces and continuous functions, it is necessarily the case that ω-limit sets exist for every $x \in X$.

Definition 2.2.3. *Let $f : X \to X$ be a continuous map. We call $A \subset X$ positively f-invariant if $f(A) \subset A$ and f-invariant if $f(A) = A$.*

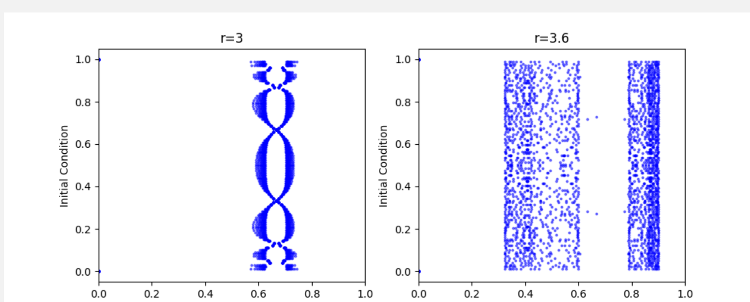
Proposition 2.2.4. *Let $f : X \to X$ be a continuous map and $x \in X$. Then $\omega(x)$ is closed and f-invariant.*

Definition 2.2.5. *Let $f : X \to X$ be a continuous map. Then a compact subset $A \subset X$ is called an attractor of f if there exists an open $U \subset X$ such that $f\left(\bar{U}\right) \subset U$ and $A = \bigcap_{i\in\mathbb{N}_0} f^i(U)$.*

- *The set U is the trapping region.*
- *The set of all points whose forward orbits converge to A is the basin of attraction, $B(A)$, which is given by*

$$B(A) = \bigcup_{n\in\mathbb{N}_0} f^{-n}(U).$$

Example 2.2.6. *We can witness the emergence of long-term dynamics by returning to the logistic map. With $r = 3$ and evolving the system for different initial conditions, we can observe how the periodic behaviour emerges. With $r = 3.6$ long-term dynamics are not prominent, although there is still some structure. More prominent patterns may appear upon transitioning to subsequences.*



Definition 2.2.7. *For $A \subset X$ and $x \in X$, the semi-Hausdorff distance is*

$$\operatorname{dist}(x, A) = \inf_{\tilde{x} \in A} d^X (x, \tilde{x}) .$$

Definition 2.2.8. *An invariant set $A \subset X$ of a continuous map $f : X \to X$ is asymptotically stable if there exists an open neighbourhood U of A such that for every $x \in U$ we have that*

$$\lim_{n \to \infty} \operatorname{dist} (f^n(x), A) = 0.$$

Proposition 2.2.9. *Attractors of continuous maps are asymptotically stable.*

2.3 Chaos

Throughout we will suppose that X is a metric space with metric d.

Definition 2.3.1. *A continuous map $f : X \to X$ has sensitive dependence if there exists a sensitivity constant $\Delta > 0$ such that for all $x \in X$ and $\epsilon > 0$, there exists a $y \in X$ with $d^X(x, y) < \epsilon$ and $n \in \mathbb{N}$ such that*

$$d^X (f^n(x), f^n(y)) \geq \Delta.$$

In other words, we have sensitive dependence if arbitrarily close initial conditions give rise to orbits that diverge by a pre-specified amount.

Definition 2.3.2. *A continuous map $f : X \to X$ is topologically transitive if for any pair of open sets $U, V \subset X$ there exists $n \in \mathbb{N}_0$ such that $f^n(U) \cap V \neq \emptyset$.*

Definition 2.3.3. *A continuous map $f : X \to X$ is chaotic if it has the following properties.*

1. *The periodic points of f are dense in X.*
2. *f is topologically transitive.*
3. *f has sensitive dependence on initial conditions.*

Theorem 2.3.4. *A continuous map on a metric space is chaotic if it has dense periodic orbits and is topologically transitive unless the metric space consists of a single periodic orbit.*

Definition 2.3.5. *A continuous map on a metric space $f : X \to X$ is topologically mixing if for any pair of non-empty open sets $U, V \subset X$ there exists $N \in \mathbb{N}$ such that for all $n > N$,*

$$f^n(U) \cap V \neq \emptyset.$$

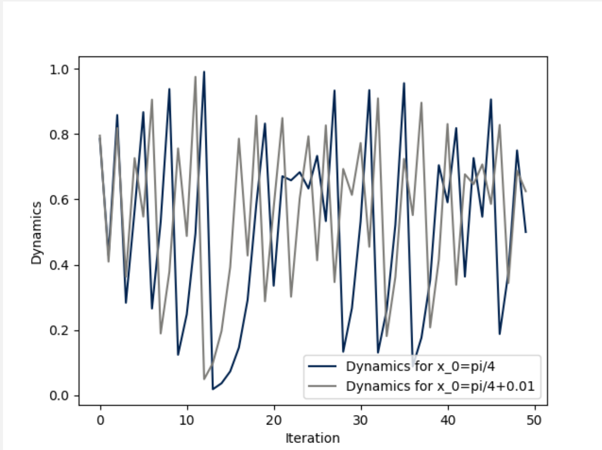
Remark 2.3.6. *Note that topological mixing is a strong property that topological transitivity.*

Theorem 2.3.7. *Every topologically mixing continuous map, on a metric space that consists of more than one point, has sensitive dependence.*

Example 2.3.8. *Consider the tent map*

$$f(x) = \begin{cases} \mu x & x < \frac{1}{2} \\ \mu(1 - x_n) & \frac{1}{2} \leq x. \end{cases}$$

For $\mu = 2$ the dynamics of the tent map is chaotic. One has dense periodic orbits on the interval $[0, 1]$, and non-periodic orbits arise if and and if the initial condition is irrational. Furthermore, we have a sensitive dependence on the initial conditions.



2.4 Topological Entropy

For a continuous map $f : X \to X$ on a compact metric space X with metric d, and $n \in \mathbb{N}$ let

$$d_n^X(x, \tilde{x}) := \max_{0 \leq k \leq n-1} d^X \left(f^k(x), f^k(\tilde{x})\right).$$

Definition 2.4.1. *Let $\epsilon > 0$.*

- *A subset $A \subset X$ is (n, ϵ)-spanning if for each $x \in X$ there is a $\tilde{x} \in A$ such that $d_n^X(x, \tilde{x}) < \epsilon$. We denote by $\operatorname{span}(n, \epsilon, f)$ the minimal cardinality of a (n, ϵ)-spanning set.*
- *A subset $A \subset X$ is (n, ϵ)-separated if for any $x \neq \tilde{x}$ we have $d_n^X(x, \tilde{x}) > \epsilon$. We denote by $\operatorname{sep}(n, \epsilon, f)$ the maximum cardinality of a (n, ϵ)-separated set.*

These quantities capture how diverse the set of orbit segments of length n are at the scale of ϵ.

Definition 2.4.2. *The topological entropy of $f : X \to X$ is defined as*

$$h_{\text{top}} := \lim_{\epsilon \to 0} \left(\limsup_{n \to \infty} \left(\frac{\log(\operatorname{span}(n, \epsilon, f))}{n} \right) \right).$$

Remark 2.4.3. *Equivalently,*

$$h_{\text{top}} = \lim_{\epsilon \to 0} \left(\limsup_{n \to \infty} \left(\frac{\log(\operatorname{sep}(n, \epsilon, f))}{n} \right) \right).$$

Topological entropy relates to the orbits of length n of a dynamical system. More specifically, topological entropy captures the exponential growth rate of the diversity of different orbit segments of length n.

Proposition 2.4.4. *Let X be a compact metric space and $f : X \to X$ a continuous map. Then any (n, ϵ)-separated set is finite, and there exists a finite (n, ϵ)-spanning set for $n \in \mathbb{N}$ and $\epsilon > 0$.*

Proposition 2.4.5. *If $f : X \to X$ is an isometry, then $h_{top}(f) = 0$.*

2.5 Topological Conjugacy

Definition 2.5.1. *Dynamical systems given by maps $f : X \to X$ and $g : Y \to Y$, are topologically conjugate if there exists a bijective homeomorphism, $h : X \to Y$, such that*

$$h \circ f = g \circ h.$$

Some properties of a dynamical system, which include those discussed so far, are preserved topological conjugacy. For example, periodic orbits of f are mapped by h to periodic orbits of g. Such properties are known as topological. The existence of h lets us view the dynamics of a system equivalently through a different perspective.

Proposition 2.5.2. *Let $f : X \to X$ and $g : Y \to Y$ be continuous and topologically conjugate.*

- *f is topologically transitive if and only if g is topologically transitive.*
- *f has dense periodic orbits if and only if g has dense periodic orbits.*
- *f is topologically mixing if and only if g is topologically mixing.*
- *f has chaotic dynamics if and only if g has chaotic dynamics.*

Proposition 2.5.3. *Let $f : X \to X$ and $g : Y \to Y$ be continuous and topologically conjugate. Then $h_{\text{top}}(f) = h_{\text{top}}(g)$.*

Definition 2.5.4. *Let $f : X \to X$ and $g : X \to X$ be continuous. Then f is an extension of g, and g is a factor of f, if there exists a surjective map $h : X \to Y$ such that*

$$h \circ f = g \circ h.$$

If h is continuous, then we say f and g are topologically semi-conjugate.

Proposition 2.5.5. *Let $: X \to X$ and $g : Y \to Y$ be topologically semi conjugate, with g being a factor of f.*

- *g has dense periodic orbits if f has dense periodic orbits.*
- *g is transitive if f is transitive.*
- *g is topologically mixing if f is topologically mixing.*
- *g is chaotic if f is chaotic. Unless Y consists of a single periodic orbit.*

Theorem 2.5.6. *Let $f : X \to X$ be topologically semi-conjugated to $g : Y \to Y$, with g being a factor of f.*

- *$h_{\text{top}}(g) \leq h_{\text{top}}(f)$.*
- *Suppose that h is such that $\sup_{y \in Y} \left(\left|h^{-1}(y)\right|\right) \leq C$ for some $C \in \mathbb{N}$. Then $h_{\text{top}}(f) = h_{\text{top}}(g)$.*

Example 2.5.7. *The tent map with $\mu = 2$ is topologically conjugate to the logistic map with $r = 4$ through*

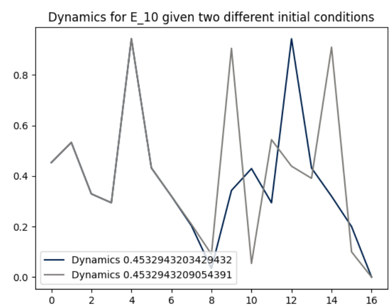
$$h(x) = \frac{2}{\pi} \arcsin\left(\sqrt{x}\right).$$

2.6 Revisiting the Motivating Example

Recall $E_k : [0, 1) \to [0, 1)$ defined by

$$E_k(x) = kx \bmod 1.$$

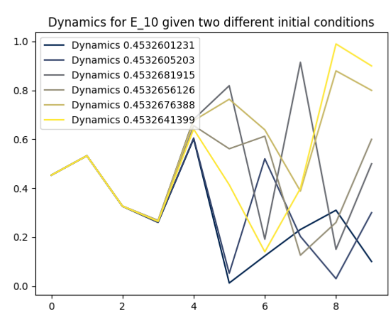
We can represent the dynamics of this system by considering x in base k and a shift operator acting on those digits. If in base k points have identical first n digits and different $(n+1)^{\text{th}}$ digit. Then their behaviour for the first n steps of the system will be identical, however, at the $(n+1)^{\text{th}}$ step the systems are at least k^{-2} apart.



Therefore, the dynamics of E_k exhibit sensitive dependence. Each open subset of $[0, 1)$, U, contains an interval of the form

$$I = \left[\sum_{i=0}^{n-1} x_i k^{-(i+1)}, k^{-n} + \sum_{i=0}^{n-1} x_i k^{-(i+1)}\right].$$

As for any $\tilde{x} \in [0, 1)$ we can find an $x \in I \subset U$ such that $E_k(x) = \tilde{x}$, it follows that, $E_k(U) = [0, 1)$ so that $E_k(U) \cap V \neq \emptyset$ for any other open subset $V \subset [0, 1)$. Therefore, the dynamics of E_k is topologically transitive. By similar arguments, the map E_k is topologically mixing.



By viewing the dynamics in base k notation, we can easily determine initial conditions whose orbits are periodic. We can simply set the digits of its base k expansion to be a recurring pattern. Moreover, we can find initial conditions that are arbitrarily close and both admit periodic orbits. We can simply fix the first n digits of the expansion, and then add the recurring pattern of digits thereafter. Such initial conditions are separated by at most k^{-n}. Suppose A_n is a set of uniformly distributed points on the circle so that the nearest-neighbour spacing is k^{-n}. Let $\epsilon > k^{-(l+1)}$ and $x \in S^1$, then there exists $y \in A_{n+l}$ such that $d^{S^1}(x, y) \leq k^{-(n+l)}$. With each application of E_k, the gap between x and y grows by a factor of most k. Therefore,

$$d^{S^1}\left(E_k^m(x), E_k^m(y)\right) \leq k^{-(n+l)+m} < \epsilon$$

so that A_{n+l} is (n, ϵ)-spanning. Now since $|A_{n+l}| = k^{n+l}$ we deduce that

$$h_{\text{top}}(E_k) = \lim_{\epsilon \to 0} \left(\limsup_{n \to \infty} \left(\frac{\log(\text{span}(n, \epsilon, f))}{n} \right) \right) \leq \lim_{\epsilon \to 0} (\log(k)) = \log(k).$$

For $\epsilon \leq k^{-(l+1)}$ one can use the same argument above to show that A_{n+l} is (n, ϵ)-separated. Therefore,

$$h_{\text{top}}(E_k) = \lim_{\epsilon \to 0} \left(\limsup_{n \to \infty} \left(\frac{\log(\text{sep}(n, \epsilon, f))}{n} \right) \right) \geq \lim_{\epsilon \to 0} (\log(k)) = \log(k).$$

From which it follows that $h_{\text{top}}(E_k) = \log(k)$. The map $h : \Sigma_k^+ \to [0, 1)$ given by

$$h(x_0 x_1 \dots) = \sum_{i=0}^{\infty} x_i k^{-(i+1)}$$

is such that

$$E_k \circ h = h \circ \sigma.$$

Thus E_k and σ are topologically semi-conjugate. We do not have topological conjugacy as h is not injective. More specifically, for $k = 10$ observe that

$$h(100 \dots) = \frac{1}{10} = \frac{\frac{9}{100}}{1 - \frac{1}{10}} = \frac{9}{10^2} + \frac{9}{10^3} + \dots = h(099 \dots).$$

3 Symbolic Dynamics

We established that we can characterise the dynamics of E_k using a shift operator. The analysis of the dynamics was easier with this alternative perspective. Therefore, in the following sections, we aim to develop the technique of symbolic dynamics. Here one establishes a connection between a dynamical system and a shift operation on a suitably defined space of symbols and analyses the dynamics through this different lens.

3.1 Topological Markov Partitions

Symbolic dynamics requires a discrete partitioning of the state space into domains. To build up the theory of this process we restrict ourselves to the one-dimensional setting of closed intervals and the circle.

Definition 3.1.1. *A map $f : X \to X$ on a compact metric space X is expanding if there exists $\epsilon > 0$ and $L > 1$ such that for all $x, \tilde{x} \in X$ with $d^X(x, \tilde{x}) < \epsilon$, we have that*

$$d^X\left(f(x), f(\tilde{x})\right) \geq L d^X(x, \tilde{x}).$$

Definition 3.1.2. *A map $f : X \to X$ on a compact matrix space X is topologically expanding if there exists some $n \in \mathbb{N}$ such that f^n is expanding.*

Throughout the discussion we let I denote a one-dimensional compact set.

Proposition 3.1.3. *A $\mathcal{C}^1$-map $f : I \to I$ is expanding if and only if $|f'(x)| \geq 1$.*

Definition 3.1.4. *A finite set of pairwise disjoint open intervals $\mathcal{R} = \{R_0, \ldots, R_{k-1}\}$ is a finite topological partition of I if*

$$I = \overline{R_0} \cup \cdots \cup \overline{R_{k-1}}.$$

Definition 3.1.5. *The refinement of a finite topological partition, $\mathcal{R}$, of I by f is given by*

$$\mathcal{R}_1 = \left\{R_{ij} = R_i \cap f^{-1}(R_j) : i, j \in \{0, \ldots, k-1\}\right\}$$

Subsequent refinements are given by

$$\mathcal{R}_m = \left\{R_{i_0 \ldots i_{m-1}} = \bigcap_{n=0}^{m-1} f^{-n}(R_{i_n}) : i_0, \ldots, i_{m-1} \in \{0, \ldots, k-1\}\right\}$$

for $m > 1$.

To represent an orbit $O_f^+(x)$ as a sequence of symbols it is necessary to record the partitions the orbit visits along its trajectory. For a finite topological partition, as given by Definition 3.1.4, the $\overline{R_i} \cap \overline{R_j}$ may not necessarily be empty. Hence, there will be ambiguity on what symbol to assign to $f(x)$ if it lies in this intersection. The refinement of a finite topological partition by f, as given by 3.1.5, resolves this issue. Note that if $x \in \overline{R_{i_0 \ldots i_{m-1}}}$ then $f^n(x) \in \overline{R_{i_n}}$ for all $n \in \{1, \ldots, m-1\}$.

Example 3.1.6. *Let us return to our motivating example. We consider the partition of S^1 into k equally sized adjacent open intervals $\mathcal{R} = \{R_0, \ldots, R_{k-1}\}$, defined by*

$$R_i = \left(\frac{i}{k}, \frac{i+1}{k}\right).$$

E_k is a contracting map, and shrinks the length of R_i by a factor of k with each application. Therefore, for $m > 1$ we have that

$$R_{i_0 \dots i_{m-1}} = \left(\sum_{n=0}^{m-1} \frac{i_n}{k^{-(n+1)}}, \sum_{n=0}^{m-1} \frac{i_n}{k^{-(n+1)}} + \frac{1}{k^m} \right).$$

It follows that

$$\lim_{m \to \infty} R_{i_0 \dots i_{m-1}} = 0.i_0 i_1 \dots \bmod 1$$

as expected by the fact that we were able to represent the dynamics as base k expansions.

Definition 3.1.7. *A continuous map $f : I \to I$ is piecewise expanding if there exists a finite topological partition $\mathcal{R} = \{R_1, \dots, R_k\}$ such that f is expanding on R_i for all $i \in \{1, \dots, k\}$.*

Example 3.1.8. *A piecewise expanding map need not be expanding. Take the tent map discussed previously with $\mu = 2$ such that*

$$f(x) = \begin{cases} 2x & x < \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \leq x. \end{cases}$$

On $[0, 1]$ the map is not expanding. However, on $\mathcal{R} = \left\{ \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right) \right\}$, we see that f is piecewise expanding as $|f'(x)| = 2 \geq 1$.

Lemma 3.1.9. *Suppose $f : I \to I$ is piecewise expanding with respect to $\mathcal{R}$. Then if $i_0 i_1 i_2 \dots \in \Sigma_k^+$ is such that $R_{i_0 \dots i_{n-1}} \neq \emptyset$ for all $n \geq 2$, then we have that*

$$\lim_{n \to \infty} \overline{R_{i_0 \dots i_{n-1}}} \in I.$$

Therefore, we have successfully been able to characterise orbits as a sequence of symbols. Moreover, we have that every sequence of symbols corresponds to the orbit of at most one initial point. However, to determine a topological semi-conjugacy we would like to identify the admissible sequences, $\Sigma_{\mathsf{adm}} \subset \Sigma_k^+$, which are those that do indeed correspond to orbits.

Definition 3.1.10. *A topological Markov chain $\Sigma_{k,A}^+$ with k symbols is a set of semi-infinite symbol sequences $i_0 i_1, \dots \in \Sigma_k^+$ characterised by rules encoded in a $k \times k$ connectivity matrix, A, where*

$$A_{ij} = \begin{cases} 1 & j \text{ is allowed to appear after } i \\ 0 & \text{otherwise.} \end{cases}$$

We endow $\Sigma_{k,A}^+$ with the naturally induced metric $d_k^{\Sigma^+}$ to form a metric space.

Remark 3.1.11.

1. *The corresponding Markov graph has nodes $0, \dots, k-1$ and a directed edge from i to j if $A_{ij} = 1$.*
2. *We will assume that all k symbols appear in the semi-infinite sequences, such that every vertex of our Markov graph has an outgoing edge.*

Definition 3.1.12. *Let $f : I \to I$ be piecewise expanding on a topological partition $\mathcal{R}$ of I. Then $\mathcal{R}$ is called* *a finite Markov partition of I if for all $i \in \{0, \ldots, k-1\}$ there exists an $S_i \subset \{0, \ldots, k-1\}$ such that*

$$\begin{cases} f(R_i) \supset R_j & j \in S_i \\ f(R_i) \cap R_j = \emptyset & j \notin S_i. \end{cases}$$

In other words, a Markov partition has the property that the closure of the image of a partition is the closure of other partitions.

Proposition 3.1.13. *Let $f : I \to I$ be piecewise expanding on a finite Markov partition $\mathcal{R}$. Then f is topologically semi-conjugated to the shift map on a topological Markov chain.*

Definition 3.1.14. *For a topological Markov chain with connectivity matrix A, such that $A_{ij} = 1$ for all $i, j \in \{0, \ldots, k-1\}$, we have $\Sigma_k^+ = \Sigma_{k,A}^+$ and the shift σ is called the full shift on k symbols.*

Example 3.1.15. *Recall the tent map defined above, and the partition that we introduced for it to be piecewise expanding. Now the closure of the image of each partition is the entire state space, so the partition is a finite Markov partition. Therefore, the tent map is a factor of the full shift on two symbols, from which it follows that the tent map is chaotic.*

Definition 3.1.16. *Let $f : I \to I$ be piecewise expanding on a compact subset $U \subset I$. Then the non-escaping set of U is defined as*

$$N(U) = \lim_{n\to\infty} \left(\bigcap_{i=0}^{n-1} f^{-i}(U) \right).$$

Remark 3.1.17.

- *Note that $N(U)$ is f-invariant.*
- *If f is piecewise expanding on U with respect to $\mathcal{R}$ then $\mathcal{R}$ is a Markov partition for $N(U)$ if it satisfies the condition of a finite Markov partition.*

Proposition 3.1.18. *Let $f : I \to I$ be piecewise expanding with respect to a finite Markov partition on $U \subset I$. Then $f|_{N(U)}$ is topologically semi-conjugated to a shift on a topological Markov chain.*

3.2 Shift Dynamics

Now that we have established conditions for topological semi-conjugacy of dynamics to symbols dynamics, we would like to understand shift maps defined on the topological Markov chains, $\sigma_A : \Sigma_{k,A}^+ \to \Sigma_{k,A}^+$. Consider $\Sigma_{k,A}^+$ endowed with the metric d^{Σ^+}. For each admissible sequence $i_0 \ldots i_{m-1}$ the cylinder set

$$C_{i_0 \ldots i_{m-1}} = \left\{ s_0 \ldots s_{m-1} s_m \cdots \in \Sigma_{k,A}^+ : i_j = s_j, j = 0, \ldots, m-1 \right\}$$

is non-empty and an open ball of radius 3^{-m+1} around each point in the set.

Proposition 3.2.1. *For a topological Markov chain $\Sigma_{k,A}^+$.*

1. *The number of distinct paths of length m on the associated Markov graph from i to j is given by $(A^m)_{ij}$.*
2. *The number of distinct paths in the Markov graph of length m starting and ending at the same vertex*

is $\operatorname{tr}(A^m)$.

Definition 3.2.2. *A topological Markov chain* $\Sigma^+_{k,A}$ *is irreducible if for all* $i, j \in \{0, \dots, k-1\}$ *there exists an* $m \geq 1$ *for which* $(A^m)_{ij} \neq 0$.

In other words, a Markov chain is irreducible if from any node we can get to any other node.

Definition 3.2.3. *A topological Markov chain* $\Sigma^+_{k,A}$ *is primitive if there exists an* $m \geq 1$ *for which* $(A^m)_{ij} \neq 0$ *for all* $i, j \in \{0, \dots, k-1\}$.

In other words, a Markov chain is primitive if there exists an $m \in \mathbb{N}$ such that there exists a path of length m between any nodes of the graph. A primitive Markov chain is an irreducible Markov chain.

Proposition 3.2.4. *The shift map on an irreducible topological Markov chain is transitive and has dense periodic orbits.*

Proposition 3.2.5. *The shift map on a topological Markov chain is transitive if and only if the topological Markov chain is irreducible.*

Corollary 3.2.6. *The shift map on a topological Markov chain is chaotic if and only if it is irreducible unless the topological Markov chain consists of a single periodic orbit.*

Proposition 3.2.7. *The shift map on a topological Markov chain is topologically mixing if and only if the topological Markov chain is primitive.*

Theorem 3.2.8. *The shift* σ_A *on the topological Markov chain* $\Sigma^+_{k,A}$ *has topological entropy*

$$h_{\text{top}}(\sigma_A) = \log(r(A))$$

where $r(A)$ *is the spectral radius of* A.

This is a positive result, as it means we can compute the topological entropy of a topological Markov chain directly from its connectivity matrix.

4 Ergodic Theory

We now consider a probabilistic view of dynamical systems rather than the topological view that we considered thus far. This will enable us to understand the behaviour of statistics along the orbits of our dynamical systems.

4.1 Invariant Probability Measures

Let $\mathcal{P}(X)$ be a set of probability measures on a measurable space $(X, \mathcal{F})$. A continuous map $f : X \to X$ induces an action $f_* : \mathcal{P}(X) \to \mathcal{P}(X)$ given by

$$f_*(\mu)(A) = \mu\left(f^{-1}(A)\right)$$

for all $A \in \mathcal{F}$.

Definition 4.1.1. *A probability measure $\mu \in \mathcal{P}(X)$ is an f-invariant probability measure if*

$$\mu(A) = f_*(\mu)(A)$$

for all $A \in \mathcal{F}$.

As entire σ-algebras are difficult to work with directly, we often focus on smaller semi-rings that generate the original σ-algebra.

Proposition 4.1.2. *Let $(X, \mathcal{F})$ be a measure space with $\mathcal{J} \subset \mathcal{F}$ a semi-ring of subsets of X that generates $\mathcal{F}$. Let $\mu \in \mathcal{P}(X)$ and $f : X \to X$ be μ-measurable. Then $\mu(A) = f_*(\mu)(A)$ for all $A \in \mathcal{J}$ if and only if $\mu(A) = f_*(\mu)(A)$ for all $A \in \mathcal{F}$.*

Theorem 4.1.3 (Krylov-Bogoliubov)**.** *Let X be a compact metric space and $f : X \to X$ be continuous. Then there exists an f-invariant Borel probability measure $\mu \in \mathcal{P}(X)$.*

For a measurable space $(X, \mathcal{F}, \mu)$ and $f : X \to X$ a μ-preserving measurable map, the tuple $(X, \mathcal{F}, \mu, f)$ denotes a measure-preserving dynamical systems. In the case where $\mu(X) < \infty$ we can normalize μ to a probability measure. Throughout the following sections, we consider the canonical setting of this where X is a compact metric space, $\mathcal{F} = \mathcal{B}(X)$, and f is continuous.

Example 4.1.4. *Let the rigid rotation map $f_a : S^1 \to S^1$ be defined by*

$$f(a) = x + a \bmod 1.$$

Then the Lebesgue measure on S^1, λ, is a f_a-invariant probability measure. One can see this by utilizing the translational invariance of the Lebesgue measure, namely

$$(f_a)_*(\lambda)(A) = \lambda\left(f_a^{-1}(A)\right) = \lambda(A - a) = \lambda(A).$$

If $a \in \mathbb{R} \setminus \mathbb{Q}$ one can show that λ is the unique measure with this property, whereas for $a \in \mathbb{Q}$ many other measures exist with this property.

4.2 Poincare Recurrence

Theorem 4.2.1 (Poincare Recurrence)**.** *Let $(X, \mathcal{F}, \mu, f)$ be a probability measure-preserving dynamical system. Let $A \in \mathcal{F}$ with $\mu(A) > 0$, then for μ-almost every $x \in A$ there exists infinitely many $i \in \mathbb{N}$ such that $f^i(x) \in A$.*

From Theorem 4.2.1, the integer

$$n_A(x) = \inf\left\{n \in \mathbb{N} : f^n(x) \in A\right\}$$

is well-defined for μ-almost all $x \in A$. However, Theorem 4.2.1 does not indicate how often the dynamics visit A.

Lemma 4.2.2 (Kac's Lemma)**.** *Let* $(X,\mathcal{F},\mu,f)$ *be a probability measure-preserving dynamical system and* $A\in\mathcal{F}$ *with* $\mu(A)>0$*. Let*

$$A^{c*}=\{x\in A^c : f^n(x)\notin A, \forall n\in\mathbb{N}\}.$$

Then, n_A *is* μ*-integrable with*

$$\int_A n_A d\mu = 1-\mu\left(A^{c*}\right).$$

4.3 Birkhoff's Ergodic Theorem

For a probability measure-preserving dynamical system $(X,\mathcal{F},\mu,f)$ let

$$\mathcal{G}:=\left\{A\in\mathcal{F} : f^{-1}(A)=A\right\}.$$

Theorem 4.3.1 (Birkhoff's Ergodic Theorem)**.** *Let* $g:X\to\mathbb{R}$ *be integrable, then*

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1} g\left(f^i(x)\right)\right)=\mathbb{E}\left(g|\mathcal{G}\right)$$

for μ*-almost all* $x\in X$*.*

What Theorem 4.3.1 tells us is that the time-average of g along $O_f^+(x)$ exists and is given by the conditional expectation of the observable g.

Definition 4.3.2. *An* f*-invariant probability measure* μ *is called ergodic if for any* f*-invariant* $A\in\mathcal{F}$ *we have that* $\mu(A)\in\{0,1\}$*.*

Example 4.3.3.

1. *The Lebesgue measure,* λ*, is ergodic for the rigid rotation map with* $a\in\mathbb{R}\setminus\mathbb{Q}$*.*
2. *For* $k>1$*, the Lebesgue measure,* λ*, is ergodic for the map* E_k*.*

Corollary 4.3.4. *Let* $g:X\to\mathbb{R}$ *be integrable and let* μ *be ergodic, then*

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1} g\left(f^i(x)\right)\right)=\mathbb{E}(g)=\int_X g\,\mathrm{d}\mu$$

for μ*-almost all* $x\in X$*.*

That is, for an ergodic measure the time-average of an observable along its forward orbits converges to the average of the observable.

Lemma 4.3.5 (Kac's Lemma for Ergodic Invariant Measures)**.** *Let* $(X,\mathcal{F},\mu,f)$ *be an ergodic probability measure preserving dynamical system, and* $A\in\mathcal{F}$ *with* $\mu(A)>0$*. Then,*

$$\int_A n_A\,\mathrm{d}\mu=1.$$

Proposition 4.3.6. *Let* $(X,\mathcal{F})$ *be a measurable space with* $f:X\to X$ *being a measurable function.*

1. *If μ_1 and μ_2 are ergodic f-invariant probability measures and $\mu_1 \ll \mu_2$ then $\mu_1 = \mu_2$.*
2. *If μ_1 and μ_2 are distinct f-invariant probability measures and $\mu = t\mu_1 + (1-t)\mu_2$ for $t \in (0,1)$, then μ is not ergodic.*
3. *Let μ_1 and μ_2 be distinct ergodic f-invariant probability measures. Then μ_1 and μ_2 are mutually singular.*

4.4 Mixing

The notion of ergodicity given by Definition 4.3.2 is limited in its capacity to distinguish the dynamics of corresponding maps. Therefore, we proceed with a different characterisation of ergodicity that captures more detail.

Proposition 4.4.1. *Let $(X, \mathcal{F}, \mu, f)$ be a probability-preserving dynamical system. Then μ is ergodic if and only if*

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} \mu\left(f^{-1}(A) \cap B\right) \right) = \mu(A)\mu(B)$$

for all $A, B \in \mathcal{F}$.

Definition 4.4.2. *Let $(X, \mathcal{F}, \mu, f)$ be a probability measure-preserving dynamical system. Then μ is mixing if*

$$\lim_{n\to\infty} \left(\mu\left(A \cap f^{-n}(B)\right)\right) = \mu(A)\mu(B)$$

for all $A, B \in \mathcal{F}$.

Therefore, mixing measures are ergodic.

Example 4.4.3. *The expanding circle map $E_k : S^1 \to S^1$, is such that the Lebesgue measure on S^1 is E_k-invariant and mixing.*

Definition 4.4.4. *Let $(X, \mathcal{F}, \mu, f)$ be a probability measure-preserving dynamical system. Then μ is weakly mixing if*

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} \left|\mu\left(f^{-1}(A) \cap B\right) - \mu(A)\mu(B)\right| \right) = 0$$

for all $A, B \in \mathcal{F}$.

It follows that mixing invariant probability measures are weakly mixing and that weakly mixing invariant probability measures are ergodic.

4.5 Markov Chain

A Markov chain generates a sequence of k symbols according to the probabilities P_{ij}, which is the probability that i follows j, under the assumption that the next symbol is only dependent on the preceding symbol. The probabilities P_{ij} combine to form a transition matrix P. Note that P is a stochastic matrix, that is

1. $P_{ij} \geq 0$ for all $i, j \in \{0, \ldots, k-1\}$, and
2. $\sum_{j=0}^{k-1} P_{ij} = 1$ for all $i \in \{0, \ldots, k-1\}$.

The connection to the connectivity matrix A of the Markov graph is given by

$$\begin{cases} A_{ij} = 0 & P_{ij} = 0 \\ A_{ij} = 1 & P_{ij} > 0. \end{cases}$$

Definition 4.5.1. *A $k \times k$ stochastic matrix P is called*

- *irreducible if for all $i, j \in \{0, \ldots, k-1\}$, there exists an $m \geq 1$ such that $(P^m)_{ij} > 0$, and*
- *primitive if there exists an $m \geq 1$ such that $(P^m)_{ij} > 0$ for all $i, j \in \{0, \ldots, k-1\}$.*

Proposition 4.5.2. *Let P be a $k \times k$ stochastic matrix.*

1. *The largest eigenvalue of P is equal to 1.*
2. *The Spectral radius of P is 1, that is $r(P) = 1$.*
3. *There exists a probability vector v such that $vP = v$.*
   - *If P is irreducible then v is unique and is a positive probability vector.*

Proposition 4.5.3. *Let P be an irreducible stochastic matrix with $v_1 P = v_1$. Then for all probability vectors $v \in \mathbb{R}^k$ we have that,*

$$\lim_{n\to\infty}\left(v\left(\frac{1}{n}\sum_{i=1}^{n-1} P^i\right)\right) = v_1,$$

In particular, if P is primitive then

$$\lim_{n\to\infty} vP^n = v_1.$$

Definition 4.5.4. *Let P be a $k \times k$ stochastic matrix with $v = (v_{i_0}, \ldots, v_{i_{n-1}})$ being one of the left probability eigenvectors for the eigenvalue 1, and let A be the associated connectivity matrix. Let $\mathcal{J}$ be the semi-ring of cylinder sets, $C_{i_0 \ldots in-1}$, of $\Sigma_{k,A}^+$ and let the pre-measure $\mu_{v,P} : \mathcal{J} \to [0,1]$ be given by*

$$\mu_{v,P}(C_{i_0 \ldots i_{n-1}}) = v_{i_0} P_{i_0 i_1} \ldots P_{i_{n-2} i_{n-1}}.$$

Then the Markov measure $\mu_{v,P} : \mathcal{B}\left(\Sigma_{k,A}^+\right) \to [0,1]$ is the unique extension of this pre-measure.

Markov measures for stochastic matrices P, where P_{ij} only depends on j, are called Bernoulli measures.

Theorem 4.5.5. *Markov measures $\mu_{v,P}$ are ergodic invariant probability measures for the shift map $\sigma_A : \Sigma_{k,A}^+ \to \Sigma_{k,A}^+$.*

4.6 Measurable Conjugacy

Having developed the connection between dynamical systems and shift maps, we want to understand how to transfer between these perspectives.

Definition 4.6.1. *Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces and $f : X \to X$ and $g : Y \to Y$ measurable maps. Then f and g are measurable conjugate if there exists a bijection $h : X \to Y$ such that $h \circ f = g \circ h$, where h and h^{-1} both measurable.*

Definition 4.6.2. *Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces and $f : X \to X$ and $g : Y \to Y$ measurable maps. Then f and g are measurably semi-conjugate if there exists a measurable surjection $h : X \to Y$ such that $h \circ f = g \circ h$.*

Example 4.6.3. *We previously mentioned the topologically conjugacy of $f(x) = 4x(x-1)$ and*

$$g(x) = \begin{cases} 2x & \frac{1}{2} > x \\ 2(1-x) & \frac{1}{2} \leq x. \end{cases}$$

through

$$h(x) = \sin^2\left(\frac{\pi}{2}x\right).$$

However, before we had h acting in the opposite direction. In fact, this conjugacy is a measurable conjugacy and one can show that $\mu : \mathcal{B}([0,1]) \to [0,1]$ defined by

$$\mu(A) = \int_A \frac{1}{\pi\sqrt{x(1-x)}}\,\mathrm{d}x$$

is an ergodic invariant measure for $f(x)$.

Proposition 4.6.4. *Let f and g be measurably (semi-)conjugated by h. Let $\mu : \mathcal{F} \to [0,\infty]$ be f-invariant, then $h_*\mu : \mathcal{G} \to [0,\infty]$ is g-invariant. Moreover, if μ is ergodic for f then $h_*\mu$ is ergodic for g.*

Generally, one is interested in the role of the Lebesgue measure on these dynamical systems.

Proposition 4.6.5. *Let $f : [0,1] \to [0,1]$ be a piecewise expanding map, topologically semi-conjugate to the shift map on an irreducible topological Markov chain, $\sigma_A : \Sigma^+_{k,A} \to \Sigma^+_{k,A}$. Then a Markov measure, $\mu_{v,P}$, for A compatible with P, induces an ergodic f-invariant Borel probability measure $h_*\mu_{v,P}$ on $[0,1]$ such that $h_*\mu_{v,P} \ll \lambda$, where λ is the Lebesgue measure, if there is a $K > 0$ such that for every $C_{i_0\dots i_{n-1}} \subset \Sigma^+_{k,A}$ we have*

$$\lambda(h(C_{i_0\dots i_{n-1}})) \geq K \cdot \mu_{v,P}(C_{i_0\dots i_{n-1}})).$$