Geometry of Curves and Surfaces*

Thomas Walker

Spring 2024

Contents

*These notes are inspired by notes written on the subject by Dr Davoud Cheraghi, and modified to complement the lecture series given on the subject by Dr Martin Taylor at Imperial College London in the spring of 2024.

1 Introduction

1.1 Curves

A curve is a smooth map $\gamma : [a, b] \to \mathbb{R}^n$, here we consider when $n \in \{2, 3\}$, where $\gamma'(t) \neq 0$ for $t \in [a, b]$. Curves in $\mathbb{R}^2$ are referred to as planar curves and curves in $\mathbb{R}^3$ are referred to as space curves.

Exercise 1.1.1. *Why do we need $\gamma'(t) \neq 0$ for $t \in [a, b]$.*

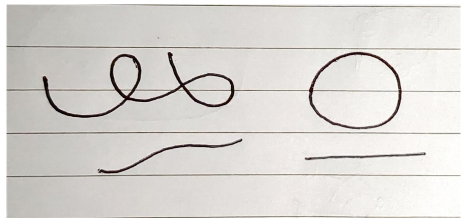
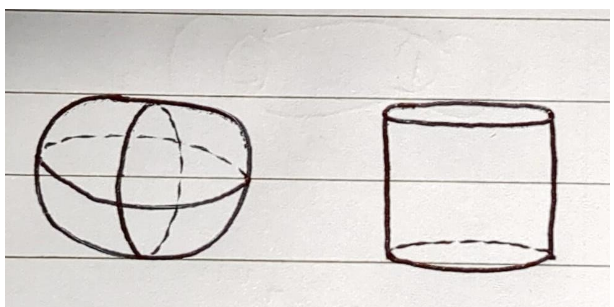


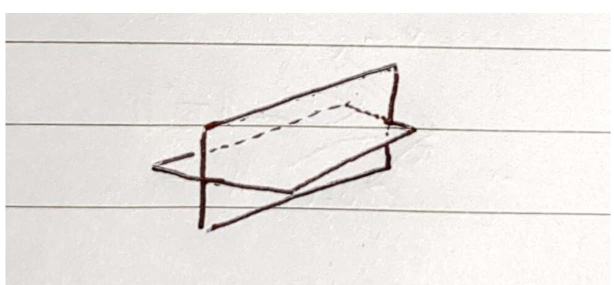
Figure 1.1.1: Examples of curves.

1.2 Surface

A subset $S \subseteq \mathbb{R}^3$ is a surface if locally it looks like $\mathbb{R}^2$.



(a) The surface on the left is the surface of a three-dimensional sphere. When the sphere has radius one this surface is denoted $\mathbb{S}^2$. The surface on the right is the surface of a cylinder. When the cylinder has radius r this surface is denoted C_r.



(b) These intersecting planes do not form a surface.

Figure 1.2.1: Examples and non-examples of surfaces.

1.3 Geometry

Geometry is the study of geometric notions such as lengths, angles, area, volumes and so forth. Topology is the study of shapes where geometric notions are not present. In geometry, a map between geometrical objects that preserves geometric notions is known as an isometry.

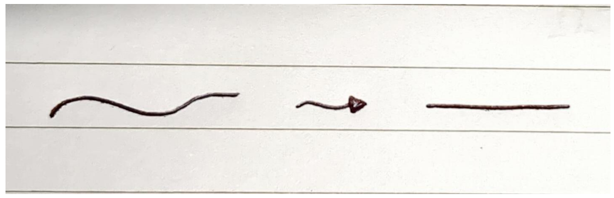


Figure 1.4.2: The extrinsic curvature of a planar curve.

1.4 Curvature

A recurring focus of study is curvature of which there are different types, namely intrinsic and extrinsic curvature. Roughly speaking, an intrinsic property is preserved under isometries, whereas an extrinsic property is dependent on the context.

Example 1.4.1.

1. *A cylinder is not intrinsically curved. One can cut down a side of a cylinder and roll it out flat whilst preserving geometric notions on the surface such as lengths, angles and areas. The process of transforming the cylinder into a plane is an example of an isometry.*
2. *A sphere has intrinsic curvature as no isometry removes its curvature. That is, it cannot be flattened without disturbing geometric properties such as length, angles and areas on the surface.*

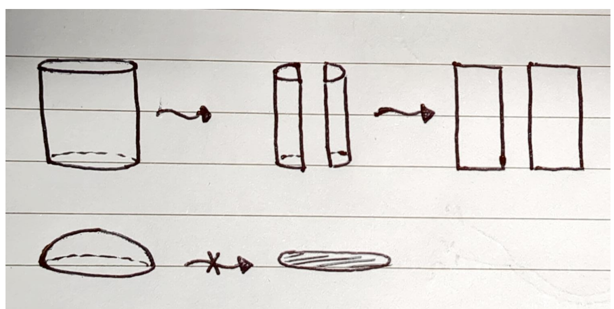


Figure 1.4.1: The extrinsic curvature of a cylinder and the intrinsic curvature of a sphere in terms of the existence of isometries.

Intrinsic curvature is absent when speaking about planar curves, every planar curve is intrinsically flat, Figure 1.4.2. Given a curve, we can stretch it out to a flat line. However, curves can exhibit extrinsic curvature. Intuitively, extrinsic planar curvature should say that a straight line has zero curvature, whereas a circle should have a non-zero constant curvature. More generally, a circle of radius r_1 should have more curvature than a circle of radius r_2 when $r_1 < r_2$. For example, we can define the curvature of a circle of radius r to be $\frac{1}{r}$, with a straight line being thought of as a circle with infinite radius. Using the idea that general planar curves can be approximated by circles, even of infinite radius, leads to the following definition of the curvature of a planar curve.

Definition 1.4.2 (Informal)**.** *A planar curve $\gamma : [a, b] \to \mathbb{R}^2$ has curvature $\kappa_\gamma : [a, b] \to \mathbb{R}$ where $\kappa_\gamma(p)$ is given by the curvature of the circle which best approximates γ at p.*

Remark 1.4.3.

1. *The curvature of a planar curve γ at a point p can be thought of as the failure of γ to be a straight line.*

2. *For space curves, we have an additional geometrical notion referred to as torsion, which can be thought of as the failure of the curve to lie in a plane.*

The curvature of a surface is more nuanced. Let $S \subseteq \mathbb{R}^3$ be a surface with $p \in S$ and n the normal to S at p. We can then consider the plane containing the normal and p. The intersection of this plane with S produces a curve for which we can talk about curvature. Therefore, with this perspective, we are considering an extrinsic curvature.

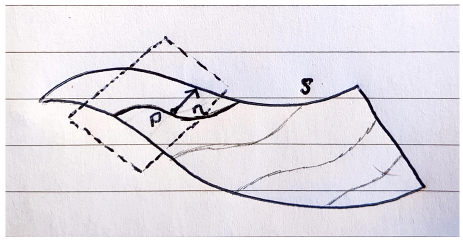


Figure 1.4.3: The curvature of the surface using an intersecting plane.

The principal curvatures $\kappa_1(p)$ and $\kappa_2(p)$ of S at p are the minimum and maximum values of the curvature of the intersecting curves between S and the planes that pass through p.

Example 1.4.4.

1. *For the surface* S^2 *we have* $\kappa_1(p) = \kappa_2(p) = 1$ *for all* $p \in \mathrm{S}^2$.
2. *For the surface* C_r *we have* $\kappa_1(p) = 0$ *and* $\kappa_2(p) = \frac{1}{r}$ *for all* $p \in C_r$.

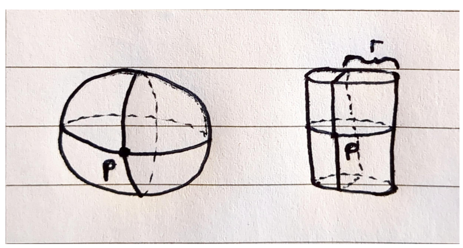


Figure 1.4.4: The intersecting planes corresponding to the principal curvatures on S^2 and C_r.

Theorem 1.4.5 (Gauss Theorema Egregium)**.** *Let* $S \subseteq \mathbb{R}^3$ *be a surface, then the quantity* $K(p) = \kappa_1(p)\kappa_2(p)$ *is an intrinsic property of* S.

Remark 1.4.6. *The function* $K_S : S \to \mathbb{R}$ *referred to in Theorem 1.4.5 is known as the Gauss curvature of* S.

Example 1.4.7.

1. *For* S^2 *we have* $K_{\mathrm{S}^2} \equiv 1$.
2. *For* C_r *we have* $K_{C_r} \equiv 0$.

Corollary 1.4.8. *The surface* S^2 *is not isometric to the plane.*

Proof. As C_r is isometric to the plane, it follows from Theorem 1.4.5 that $K_P \equiv 0$, where P is a plane. Therefore,

$K_P \neq K_{\mathrm{S}^2}$ and so S^2 is not isometric to P by Theorem 1.4.5. □

1.5 Geodesics

Geodesics are curves in a surface S that are as straight as possible. For example, on the cylinder, a geodesic should have the property that under an isometry to the plane they map to straight lines. On S^2, all geodesics are great circles, which are circles on a plane which passes through the origin.

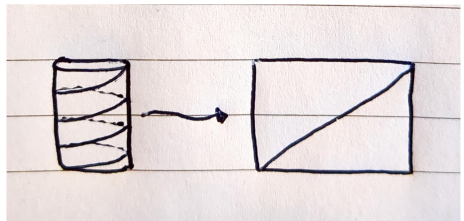


Figure 1.5.1: An illustration of how the geodesic of a cylinder transforms into a straight line under a cut and roll isometry.

1.6 Geometry and Topology

Questions between geometry and topology arise naturally and are important in many applications such as solving partial differential equations. A question one may ask is whether we can embed S^2, as a topological space, into $\mathbb{R}^3$ such that the Gauss curvature of the resulting surface is non-positive. Similarly, one may ask if there is an embedding of S^2 into $\mathbb{R}^3$ which contains non-intersecting geodesics.

Theorem 1.6.1 (Gauss-Bonnet). *Let $S \subseteq \mathbb{R}^3$ be a compact orientable surface. Then*

$$\int_S K \, \mathrm{d}A_S = 2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic of S as a topological surface.

Example 1.6.2. *For S^2 we have $\chi\left(\mathrm{S}^2\right) = 2$. Therefore,*

$$\int_{\mathrm{S}^2} K \, \mathrm{d}\mathrm{S}^2 = 4\pi > 0.$$

Therefore, S^2 cannot be embedded into $\mathbb{R}^3$ such that $K \leq 0$.

1.7 Solution to Exercises

Exercise 1.1.1

Solution. Ensuring $\gamma'(t) \neq 0$ for all $t \in [a, b]$ means that the tangent at any point along the curve is well-defined. More specifically, for a curve $\gamma : [a, b] \to \mathbb{R}^n$, the tangent to the curve at the point $\gamma(t_0)$ is given by

$$s \mapsto \gamma(t_0) + s\gamma'(t_0).$$

□

2 Curves

2.1 Regular Curves

The map $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ will be the Euclidean dot product, that is

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

The Euclidean norm will thus be

$$|x| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2},$$

which provides a notion of length for $\mathbb{R}^n$.

Definition 2.1.1. *A parameterised curve in $\mathbb{R}^n$ is a smooth map $\gamma : I \to \mathbb{R}^n$, where $I \subseteq \mathbb{R}$ is an interval. Moreover, γ is regular if $|\gamma'(t)| \neq 0$ for all $t \in I$.*

Remark 2.1.2.

1. *Here we will only be concerned with $n = 2$ and $n = 3$, which we refer to as a planar curve and a space curve respectively.*
2. *Note that a curve can either refer to a regular parameterisation $\gamma : I \to \mathbb{R}^n$ or the image of a regular parameterisation, namely $\gamma([a, b]) \subseteq \mathbb{R}^n$. The former interpretation is akin to thinking of the curve as the trajectory of a particle that is always in motion. Whilst the latter is a subset of $\mathbb{R}^n$ whose properties such as length we will be interested in.*

Example 2.1.3. *Examples of regular curves include the following.*

1. *A straight line segment given by $\gamma_1 : [0, 1] \to \mathbb{R}^2$ where $\gamma_1(t) = (2t - 1, 3t + 2)$.*
2. *A circle with radius $r > 0$ given by $\gamma_2 : [0, 2\pi) \to \mathbb{R}^2$ where $\gamma_2(t) = (r\cos(t), r\sin(t))$.*
3. *A helix given by $\gamma_3 : [0, 6\pi] \to \mathbb{R}^3$ where $\gamma_3(t) = (\cos(t), \sin(t), t)$.*
4. *Curves can be self-intersecting, $\gamma_4 : \mathbb{R} \to \mathbb{R}^2$ where $\gamma_4(t) = \left(t^3 - 4t, t^2 - 4\right)$.*
5. *The Folium of Descartes is the curve $\gamma_5 : (-1, \infty) \to \mathbb{R}^2$ where $\gamma_5(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$. This curve is injective but not homeomorphic to its image.*

An example of a non-regular curve is the following.

6. *The curve $\gamma_6 : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma_6(t) = \left(t^3, t^2\right)$ is a parameterised curve but it is not regular as $\gamma_6'(0) = (0, 0)$. Note that $\gamma_6(\mathbb{R}) = \left\{(x, y) \in \mathbb{R}^2 : y = x^{\frac{2}{3}}\right\}$ has a cusp at the origin.*

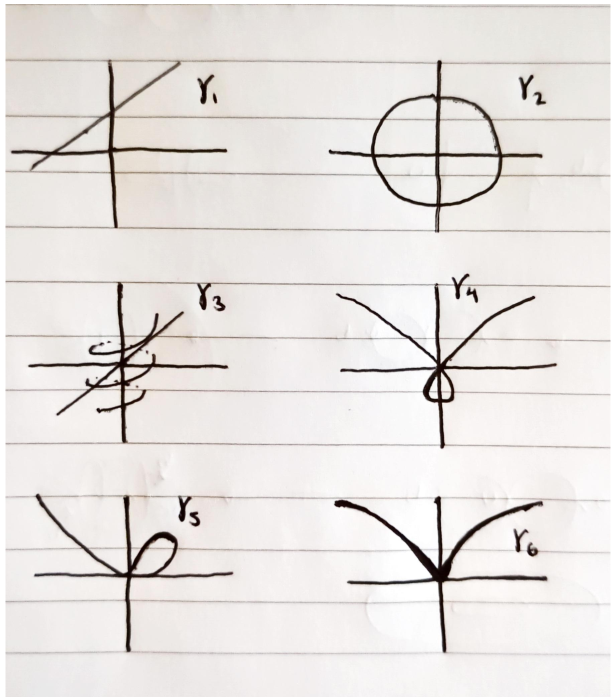


Figure 2.1.1: Illustrations of the curves stated in Example 2.1.3

Definition 2.1.4. *Let γ represent a regular curve. Then the tangent line to γ at $\gamma(t_0)$ is given by*

$$\{\gamma(t_0) + \gamma'(t_0)s : s \in \mathbb{R}\}.$$

Example 2.1.5. *Being smooth is not sufficient to ensure that a tangent line is well-defined. Consider $\alpha : (-1,1) \to \mathbb{R}^3$ given by*

$$\alpha(t) = \begin{cases} \left(\exp\left(-\frac{1}{t^2}\right), \exp\left(-\frac{1}{t^2}\right), 0\right) & t > 0 \\ (0,0,0) & t = 0 \\ \left(-\exp\left(-\frac{1}{t^2}\right), \exp\left(-\frac{1}{t^2}\right), 0\right) & t < 0. \end{cases}$$

Then α is smooth, however, the tangent line to α at $\alpha(0)$ is not well-defined as $\alpha'(0) = (0,0,0)$.

Definition 2.1.6. *Let $J, I \subseteq \mathbb{R}$ be intervals. If $f : J \to I$ is a diffeomorphism, that is $f'(t) \neq 0$ for every $t \in J$, and $\gamma : I \to \mathbb{R}^n$ is a regular curve then $\gamma \circ f : J \to \mathbb{R}^n$ is a reparameterisation of γ.*

Remark 2.1.7. *A reparameterisation of a regular curve as given in Definition 2.1.6, is a regular curve. Indeed,*

$$|\gamma'(f(t))| = |\gamma'(f(t))|\,|f'(t)| \neq 0.$$

Most properties of interest are invariant under reparameterisation.

2.2 Length

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a regular curve and consider the partition

$$a = t_0 < \cdots < t_n = b.$$

Then the length of γ denoted $L(\gamma)$, is approximated by

$$L(\gamma) \approx \sum_{i=0}^{n-1} |\gamma(t_{i+1}) - \gamma(t_i)| \approx \sum_{i=0}^{n-1} |\gamma'(t_i)|\, |t_{i+1} - t_i|,$$

where the second approximation comes from the mean value theorem. The approximation becomes better as the partition becomes finer.

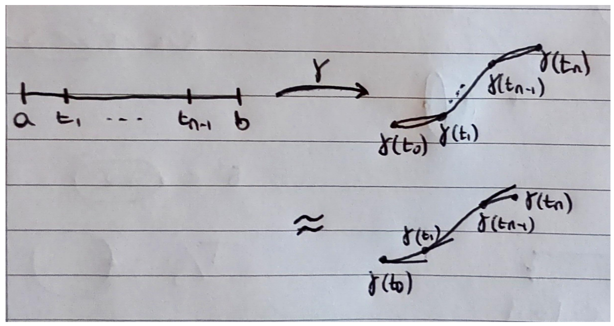


Figure 2.2.1: Approximating the length of a curve by using line segments between points of a partition.

Definition 2.2.1. *The length of a regular curve $\gamma : [a, b] \to \mathbb{R}^n$ is given by*

$$L(\gamma) = \int_a^b |\gamma'(t)| \, \mathrm{d}t.$$

Example 2.2.2. *Using Definition 2.2.1 we can find the length of a few of the curves from Example 2.1.3.*

1. $L(\gamma_1) = \int_0^1 |(2, 3)| \, \mathrm{d}t = \sqrt{13}$.
2. $L(\gamma_2) = \int_0^{2\pi} |(-r\sin(t), r\cos(t))| \, \mathrm{d}t = \int_0^{2\pi} r \, \mathrm{d}t = 2\pi r$.
3. $L(\gamma_3) = \int_0^{6\pi} |(-\sin(t), \cos(t), 1)| \, \mathrm{d}t = 6\sqrt{2}\pi$.

Proposition 2.2.3. *The length of a regular curve $\gamma : I \to \mathbb{R}^n$ is independent of the parameterisation. That is,*

$$L(\gamma) = L(\gamma \circ f),$$

where f is a reparameterisation as given by Definition 2.1.6.

Proof. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a regular curve, with $f : [c, d] \to [a, b]$ a smooth function such that $f'(s) \neq 0$ for all $s \in [c, d]$ and $f(\{c, d\}) = \{a, b\}$. Then $\psi := \gamma \circ f : [c, d] \to \mathbb{R}^n$ is a reparameterisation of $\gamma([a, b])$. Since $f'(s) \neq 0$ for all $s \in [c, d]$ and f' is continuous on $[c, d]$ it follows that $f' > 0$ on $[c, d]$ or $f' < 0$ on $[c, d]$. Without loss of generality suppose that $f' > 0$ on $[c, d]$ such that $f(c) = a$ and $f(d) = b$. With $\gamma(t) = (x_1(t), \dots, x_n(t))$ we have

$$\begin{aligned}
|\psi'(s)| &= |(\gamma \circ f)'(s)| \\
&= |((x_1 \circ f)'(s), \dots, (x_n \circ f)'(s))| \\
&= |(x_1'(f(s))f'(s), \dots, x_n'(f(s))f'(s))| \\
&= |f'(s)|\, |(x_1'(f(s)), \dots, x_n'(f(s)))| \\
&= f'(s)\, |\gamma'(f(s))| \, .
\end{aligned}$$

Therefore,

$$\begin{aligned}
L(\psi([c,d])) &= \int_c^d |\psi'(s)| \,\mathrm{d}t \\
&= \int_c^d f'(s)\,|\gamma'(f(s))| \,\mathrm{d}t \\
&\overset{t=f(s)}{=} \int_{f(c)}^{f(d)} |\gamma'(t)| \,\mathrm{d}t \\
&= \int_a^b |\gamma'(t)| \,\mathrm{d}t \\
&= L(\gamma([a,b])).
\end{aligned}$$

□

Definition 2.2.4. *A regular curve $\gamma : [a,b] \to \mathbb{R}^n$ is parameterised by arc length if $|\gamma'(t)| = 1$ for all $t \in (a,b)$.*

Remark 2.2.5.

1. *With the interpretation that a curve traces the trajectory of a non-stationary particle, an arc length parameterisation ensures that the particle moves at a unit speed along the curve. Consequently, the arc length parameterisation is also referred to as the unit speed parameterisation.*
2. *Note that if a regular curve $\gamma : [a,b] \to \mathbb{R}^n$ is parameterised by arc length, then $L(\gamma) = b - a$.*
3. *The arc-length parameterisation of a regular curve is not unique. In particular, one can reverse the direction of an arc-length parameterisation and obtain a different arc-length parameterisation.*

Proposition 2.2.6. *Any regular curve $\gamma : [a,b] \to \mathbb{R}^n$ can be parameterised by arc length.*

Proof. Let $\gamma : [a,b] \to \mathbb{R}^n$ be a regular curve. Let $h : [a,b] \to \mathbb{R}^n$ be given by

$$t \mapsto \int_a^t |\gamma'(u)| \,\mathrm{d}u + c.$$

As $\gamma'(u)$ exists and is continuous for all $u \in [a,b]$, the function h is well-defined with $h(a) = c$ and $h(b) = L(\gamma([a,b])) + c =: d$. Moreover,

$$h'(t) = |\gamma'(t)|\,, \tag{2.2.1}$$

which implies that h has a non-zero derivative at every point $t \in [a,b]$. In particular, this means that $h : [a,b] \to [c,d]$ has a well-defined inverse. Let $f : [c,d] \to [a,b]$ be the inverse of $h : [a,b] \to [c,d]$, which we note is smooth with a non-zero derivative. Thus we can consider the reparameterisation $\psi := \gamma \circ f : [c,d] \to \mathbb{R}^n$. As $(h \circ f) = t$, for $t \in [c,d]$ we have $h'(f(t))f'(t) = 1$, and so as the derivatives are non-zero we deduce that

$$|f'(t)| = \frac{1}{|h'(f(t))|}. \tag{2.2.2}$$

Using (2.2.1) and (2.2.2) it follows that

$$|\psi'(t)| = |\gamma'(f(t))f'(t)| = |h'(f(t))|\,\frac{1}{|h'(f(t))|} = 1.$$

Therefore, ψ is an arc length parameterisation of $\gamma : [a,b] \to \mathbb{R}^n$. □

2.3 Curvature

Recall that regular curves are intrinsically flat, that is they are locally isometric to a straight line. Therefore, the curvature of a curve refers to an extrinsic property.

Definition 2.3.1. *Let $\gamma : [0, l] \to \mathbb{R}^n$ be a regular curve parameterised by arc length. The curvature of γ at $\gamma(t)$ is*

$$k(t) := |\gamma''(t)| .$$

The curvature vector is given by $\mathbf{k}(t) = \gamma''(t)$.

Lemma 2.3.2. *The curvature and curvature vector of a regular curve are independent of the arc length parameterisation of the curve.*

Proof. Let γ be a regular curve parameterised by arc length. Let $\psi = \gamma \circ f$ be an arc length parameterisation of γ. Then

$$1 = |\psi'(t)| = |\gamma'(f(t))| \, |f'(t)| = |f'(t)| .$$

Thus,

$$f(t) = \pm t + c$$

for some constant c. Therefore, $\psi(t) = \gamma(\pm t + c)$ and so

$$\psi''(t) = \gamma''(t).$$

□

Example 2.3.3. *Let $\gamma : [0, 2\pi) \to \mathbb{R}^2$ be given by*

$$\gamma(t) = (R\cos(t), R\sin(t)).$$

Note that $|\gamma'(t)| = R$ and thus γ is not parameterised by arc length. Let

$$h(t) := \int_0^t |\gamma'(s)| \, \mathrm{d}s = tR,$$

and set

$$f(t) := \frac{t}{R}.$$

Then with

$$\psi(t) = (\gamma \circ f)(t) = \left(R\sin\left(\frac{t}{R}\right), R\sin\left(\frac{t}{R}\right)\right)$$

we note that $|\psi'(t)| = 1$, and so ψ is parameterised by arc length. Hence, we can write

$$\mathbf{k}(t) = \left(-\frac{1}{R}\cos\left(\frac{t}{R}\right), -\frac{1}{R}\sin\left(\frac{t}{R}\right)\right)$$

and $k(t) = \frac{1}{R}$. This corresponds to our osculating circle heuristic developed previously.

Proposition 2.3.4. *Let $\gamma : I \to \mathbb{R}^n$ be a regular curve parameterised by arc length. Then $k(t) = 0$ for all $t \in I$ if and only if γ is a straight line.*

Proof. Note that $k \equiv 0$ if and only if $\gamma'' \equiv 0$ which happens if and only if $\gamma = at + b$ for some $a, b \in \mathbb{R}^n$. □

Remark 2.3.5. *The curve must be parameterised by arc length in Proposition 2.3.4. Consider $\gamma : [0,1] \to \mathbb{R}^2$ given by $\gamma(t) = (0, e^t)$. Then $\gamma([0,1])$ is a straight line, however,*

$$\gamma''(t) = \left(0, e^t\right) \neq (0,0).$$

Indeed,

$$|\gamma'(t)| = e^t,$$

which is not equal to one for all $t \in [0,1]$.

Proposition 2.3.6. *Let $\gamma : [a,b] \to \mathbb{R}^n$ be a regular curve parameterised by arc length. For every $t \in [a,b]$, the curvature vector $\mathbf{k}(t)$ is perpendicular to the tangent vector $\gamma'(t)$ to the curve $\gamma([a,b])$ at $\gamma(t)$.*

Proof. As γ is parameterised by arc length we have $\langle \gamma'(t), \gamma'(t) \rangle = 1$ for all $t \in [a,b]$, thus $\frac{\mathrm{d}}{\mathrm{d}t} \langle \gamma'(t), \gamma'(t) \rangle = 0$ for all $t \in [a,b]$. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \gamma'(t), \gamma'(t) \rangle = \langle \gamma''(t), \gamma'(t) \rangle + \langle \gamma'(t), \gamma''(t) \rangle = 2 \langle \mathbf{k}(t), \gamma'(t) \rangle .$$

Therefore, $\langle \mathbf{k}(t), \gamma'(t) \rangle = 0$ for all $t \in [a,b]$ which means that $\mathbf{k}(t)$ is perpendicular to the the tangent line of $\gamma([a,b])$ at $\gamma(t)$. $\square$

Remark 2.3.7.

1. *For planar curves, the curvature vector can point in one of two directions whilst still being orthogonal to the tangent vector, Figure 2.3.1. The direction of the curvature vector points depends on the direction in which the arc length parameterisation is given.*

2. *Returning to the interpretation that a curve is the trajectory of a particle, we noted that with an arc length parameterisation, the particle travels at unit speed along this trajectory. Thus, acceleration is only experienced in the centripetal direction, that is the direction orthogonal to its motion. Hence, the curvature gives the acceleration of the particle and can be thought of as being responsible for changing the direction of the curve.*

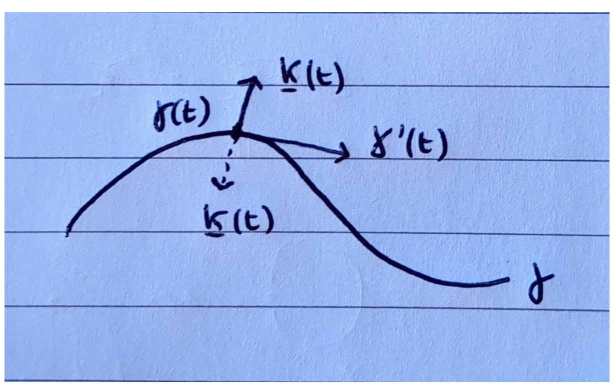


Figure 2.3.1: The tangent vector and curvature vector for a regular curve in $\mathbb{R}^2$.

Exercise 2.3.8. *Let $\gamma : (a,b) \to \mathbb{R}^2$ be a regular curve parameterised by arc length, with $k_\gamma(t_0) > 0$ for some $t_0 \in (a,b)$. For $v \in \mathbb{R}^2$, let*

$$C_v := \left\{ p \in \mathbb{R}^2 : |p - v| = |\gamma(t_0) - v| \right\}$$

and let $f_v : (a,b) \to \mathbb{R}^2$ be given by

$$f_v(t) = |\gamma(t) - v|^2.$$

1. *Show that C_v is tangent to γ at t_0 if and only if $f_v'(t_0) = 0$.*
2. *Show that there exists a $v \in \mathbb{R}^2$ such that $f_v'(t_0) = f_v''(t_0) = 0$ with C_v having a radius of $\frac{1}{k_\gamma(t_0)}$.*

Remark 2.3.9. *Note that C_v from Exercise 2.3.8 is a circle that passes through $\gamma(t_0)$. Thus, statement 2 of Exercise 2.3.8 says that the tangential circle that best approximates the curve γ at t_0 has radius $\frac{1}{k_\gamma(t_0)}$, which relates to our osculating circle heuristic.*

Proposition 2.3.10. *Let $\gamma : [a,b] \to \mathbb{R}^3$ be a regular curve parameterised by arc length, and suppose that $|\gamma(t)|$ achieves its maximum value at some $t_0 \in (a,b)$. Then,*

$$|k(t_0)| \geq \frac{1}{|\gamma(t_0)|}.$$

Proof. Consider the function $f : [a,b] \to \mathbb{R}$ given by

$$f(t) = \langle \gamma(t), \gamma(t) \rangle.$$

Then

$$f'(t) = 2 \langle \gamma(t), \gamma'(t) \rangle$$

and

$$\begin{aligned} f''(t) &= 2 \langle \gamma(t), \gamma''(t) \rangle + 2 \langle \gamma'(t), \gamma'(t) \rangle \\ &= 2 \langle \gamma(t), \gamma''(t) \rangle + 2 \end{aligned}$$

where the second equality follows as γ is parameterised by arc length. Then as $f(t_0)$ is a maximum, and t_0 is an interior point of (a,b), it follows that $f''(t_0) \leq 0$. Thus,

$$2 \langle \gamma(t_0), \gamma''(t_0) \rangle + 2 \leq 0. \tag{2.3.1}$$

In particular, it must be the case that $\langle \gamma(t_0), \gamma''(t_0) \rangle < 0$, and so

$$\langle \gamma(t_0), \gamma''(t_0) \rangle = - |\langle \gamma(t_0), \gamma''(t_0) \rangle| .$$

Thus, (2.3.1) becomes

$$1 \leq |\langle \gamma(t_0), \gamma''(t_0) \rangle| .$$

Using the Cauchy-Schwartz inequality it follows that

$$1 \leq |\gamma(t_0)| \, |\gamma''(t_0)| = |\gamma(t_0)||k(t_0)|.$$

Upon rearrangement we arrive at

$$|k(t_0)| \geq \frac{1}{|\gamma(t_0)|}.$$

□

2.4 Frenet Frames

Definition 2.4.1. *Let $\gamma : [a,b] \to \mathbb{R}^3$ be a regular curve parameterised by arc length for which $k_\gamma(t) \neq 0$ for every $t \in [a,b]$.*

1. *The unit tangent vector to γ at $t \in [a,b]$ is $T(t) := \gamma'(t)$.*
2. *The principal normal vector to γ at $t \in [a,b]$ is $N(t) := \frac{T'(t)}{|T'(t)|}$.*

3. *The binormal vector to γ at $t \in [a, b]$ is $B(t) := T(t) \times N(t)$.*

The triple $(T(t), N(t), B(t))$ for $t \in [a, b]$ is a positively oriented orthonormal basis for $\mathbb{R}^3$ referred to as the Frenet frame for γ at t.

Remark 2.4.2.

1. *The principal normal vector of γ is well-defined as $T' \neq 0$ by the assumption on γ. Moreover, as $T'(t) = \mathbf{k}(t)$, it follows by Proposition 2.3.6 that the principal normal vector is orthogonal to the tangent vector.*

2. *In three dimensions there are infinitely many normal directions to the tangent vector. The principal normal vector is chosen canonically such that $N(t)$ lies in the osculating plane of γ at t. The osculating plane is well-defined when γ is not a straight line and so that is why we require that $k_\gamma(t) \neq 0$.*

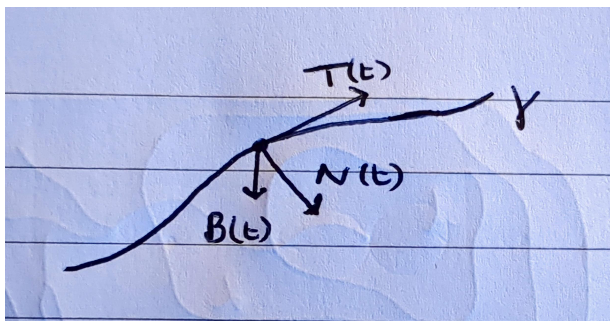


Figure 2.4.1: The Frenet frame for a curve in $\mathbb{R}^3$.

Proposition 2.4.3. *Let $\gamma : (a, b) \to \mathbb{R}^3$ be a regular curve parameterised by arc length, with $|\gamma''(t)| \neq 0$ for all $t \in (a, b)$. Then the following statements hold.*

1. *$T'(t) = k(t)N(t)$.*

2. *There exists a function $\tau : (a, b) \to \mathbb{R}$ such that*

$$B'(t) = -\tau(t)N(t)$$

and

$$N'(t) = \tau(t)B(t) - k(t)T(t).$$

Proof.

1. $T'(t) = \gamma''(t) = \mathbf{k}_\gamma(t) = k(t)N(t)$ by Definition 2.4.1.

2. As $\langle B(t), B(t)\rangle = 1$ it follows that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle B(t), B(t)\rangle = 2\langle B'(t), B(t)\rangle .$$

Moreover,

$$\begin{aligned} B'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(T(t) \times N(t)) \\ &= T'(t) \times N(t) + T(t) \times N'(t) \\ &= T(t) \times N'(t). \end{aligned}$$

Hence, $B'(t)$ is orthogonal to $B(t)$ and $T(t)$. This implies that $B(t)$ is proportional to $N(t)$ and so there exists a $\tau : (a, b) \to \mathbb{R}$ such that

$$B'(t) = -\tau(t)N(t).$$

In particular, as $N(t) = B(t) \times T(t)$ we deduce that

$$\begin{aligned} N'(t) &= B'(t) \times T(t) + B(t) \times T'(t) \\ &= -\tau(t)N(t) \times T(t) + k(t)B(t) \times N(t) \\ &= -\tau(t)\left(-B(t)\right) + k(t)\left(-T(t)\right) \\ &= \tau(t)B(t) - k(t)T(t). \end{aligned}$$

□

Remark 2.4.4.

1. *The equations of Proposition 2.4.3, also referred to as the Frenet equations, can be written in matrix form as*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

2. *The function $\tau = \tau_\gamma$ is referred to as the torsion on γ. Just as curvature can be interpreted as a measure of the failure of a curve to be a straight line, torsion can be interpreted as a measure of the failure of a curve to be planar.*

3. *From the Frenet equations it follows that torsion and curvature completely govern the Frenet frame.*

Proposition 2.4.5. *Let $\gamma : (a, b) \to \mathbb{R}^3$ be a regular curve parameterised by arc length, with $|\gamma''(t)| \neq 0$ for all $t \in (a, b)$. Then γ is contained in a plane if and only if $\tau \equiv 0$.*

Proof. ($\Rightarrow$). There exists a unit vector $\mathbf{v} \in \mathbb{R}^3$ and $d \in \mathbb{R}$ such that $\langle \gamma(t), \mathbf{v} \rangle = d$ for all $t \in [a, b]$. Hence, $\langle \gamma'(t), \mathbf{v} \rangle = 0$ for all $t \in [a, b]$ and so

$$\langle T(t), \mathbf{v} \rangle = 0$$

for all $t \in [a, b]$. Again this means that $\langle T'(t), \mathbf{v} \rangle = 0$ for all $t \in [a, b]$ and so

$$\langle k(t)N(t), \mathbf{v} \rangle = 0$$

for all $t \in [a, b]$. Since $k(t) = |\gamma''(t)| \neq 0$ it follows that

$$\langle N(t), \mathbf{v} \rangle = 0$$

for all $t \in [a, b]$. Thus, $\mathbf{v}$ is orthogonal to both $T(t)$ and $N(t)$. Therefore, as $\mathbf{v}$ is a unit vector we have that $B(t) = \pm\mathbf{v}$ which implies that $0 = B'(t) = -\tau(t)N(t)$ and so $\tau \equiv 0$.
($\Leftarrow$). It follows that $B' = -\tau N \equiv 0$ and so $B(t) = \mathbf{c}$ for some $\mathbf{c} \in \mathbb{R}^3$. Observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma(t), \mathbf{c} \rangle = \langle \gamma'(t), \mathbf{c} \rangle = \langle T(t), B(t) \rangle \overset{(1)}{=} 0,$$

where in (1) we have used the fact that $B(t) = T(t) \times N(t)$. Therefore, $\langle \gamma(t), \mathbf{c} \rangle = d$ for some $d \in \mathbb{R}$, and so

$$\gamma(t) \in \left\{(x, y, z) \in \mathbb{R}^3 : \langle (x, y, z), \mathbf{c} \rangle = d\right\}$$

which is the plane perpendicular to the vector $\mathbf{c} = B(t)$. Thus, γ is contained in the plane spanned by T and N. □

Example 2.4.6.

1. *Let* $\gamma_1 : [0, 2\pi) \to \mathbb{R}^3$ *be given by*
$$\gamma_1(t) = (\cos(t), \sin(t), 0).$$
Then
$$T(t) = (-\sin(t), \cos(t), 0)$$
and so
$$N(t) = (-\cos(t), -\sin(t), 0).$$
Thus,
$$B(t) = T(t) \times N(t) = (0, 0, 1).$$
In this case,
$$k(t) = |T'(t)| = 1,$$
and $\tau \equiv 0$ *as expected by Proposition 2.4.5.*

2. *Let* $\gamma_2 : \mathbb{R} \to \mathbb{R}^3$ *be given by*
$$\gamma_2(t) = \left(\cos\left(\frac{t}{\sqrt{2}}\right), \sin\left(\frac{t}{\sqrt{2}}\right), \frac{t}{\sqrt{2}}\right).$$
Note that
$$|\gamma_2'(t)| = \frac{1}{\sqrt{2}}\left|\left(-\sin\left(\frac{t}{\sqrt{2}}\right), \cos\left(\frac{t}{\sqrt{2}}\right), 1\right)\right| = 1,$$
and so γ *is parameterised by arc length. Then*
$$T(t) = \frac{1}{\sqrt{2}}\left(-\sin\left(\frac{t}{\sqrt{2}}\right), \cos\left(\frac{t}{\sqrt{2}}\right), 1\right)$$
and so
$$N(t) = \left(-\cos\left(\frac{t}{\sqrt{2}}\right), -\sin\left(\frac{t}{\sqrt{2}}\right), 0\right).$$
Thus,
$$B(t) = T(t) \times N(t) = \frac{1}{\sqrt{2}}\left(\sin\left(\frac{t}{\sqrt{2}}\right), -\cos\left(\frac{t}{\sqrt{2}}\right), 1\right)$$
from which it follows that $B'(t) = -\frac{1}{2}N(t)$*. In this case* $k(t) \equiv \frac{1}{2}$ *and* $\tau \equiv \frac{1}{2}$*.*

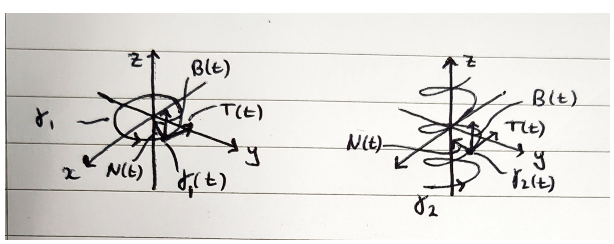


Figure 2.4.2: The curves of Example 2.4.6. Note how the basis vectors of the Frenet frame are drawn with the origin being the corresponding point on the curve at time t.

Definition 2.4.7. *Let* $\gamma : [a, b] \to \mathbb{R}^3$ *be a regular curve parameterised by arc length. Then a rigid motion of* γ *in* $\mathbb{R}^3$ *is a regular curve* $\psi : [a, b] \to \mathbb{R}^3$ *such that there exists a* $A \in \mathrm{SO}(3)$ *and a* $\mathbf{c} \in \mathbb{R}^3$ *with* $\psi = A\gamma + \mathbf{c}$*.*

Lemma 2.4.8. *For a regular curve parameterised by arc length, its arc length, curvature and torsion are preserved under rigid motion.*

Proof. Let $\gamma : [a,b] \to \mathbb{R}^3$ be a regular curve parameterised by arc length. Let $A \in \mathrm{SO}(3)$ and $\mathbf{c} \in \mathbb{R}^3$. Let $\psi : [a,b] \to \mathbb{R}^3$ be given by $\psi = A\gamma + \mathbf{c}$. It follows that

$$|\psi'| = \left|(A\gamma + \mathbf{c})'\right| = |A\gamma'| = |\gamma'| = 1$$

and so ψ is parameterised by arc length. Moreover,

$$k_\psi = |\psi''| = |(A\gamma + \mathbf{c})''| = |A\gamma''| = |\gamma''| = k_\gamma$$

and so ψ has the same curvature as γ. Assume that $\gamma''(t) \neq 0$ for all $t \in [a,b]$ such that τ_γ is well-defined. Note that $T_\psi = AT_\gamma$ and $N_\psi = AN_\gamma$ so that

$$\begin{aligned} B_\psi &= T_\psi \times N_\psi \\ &= AT_\gamma \times AN_\gamma \\ &= A\left(T_\gamma \times N_\gamma\right) \\ &= AB_\gamma. \end{aligned}$$

As $B'_\psi = -\tau_\psi N_\psi$ we have that

$$\begin{aligned} AB'_\gamma &= (AB_\gamma)' \\ &= (B_\psi)' \\ &= -\tau_\psi N_\psi \\ &= -\tau_\psi \left(AN_\gamma\right) \\ &= A\left(-\tau_\psi N_\gamma\right) \end{aligned}$$

which implies that $AB'_\gamma = A\left(-\tau_\psi N_\gamma\right)$. As A is invertible we deduce that $B'_\gamma = -\tau_\psi N_\gamma$ which implies that $\tau_\gamma N_\gamma = \tau_\psi N_\gamma$ and so $\tau_\psi = \tau_\gamma$. ☐

Proposition 2.4.9. *Let $I \subseteq \mathbb{R}$ be an interval and let $A : I \to \mathbb{R}^{n\times n}$ be smooth. Then for any $X_0 \in \mathbb{R}^{n\times n}$ there exists a unique function $X : I \to \mathbb{R}^{n\times n}$ such that*

$$\dot{X}(t) = A(t)X(t)$$

for all $t \in I$ and $X(t_0) = X_0$ for fixed $t_0 \in I$.

Theorem 2.4.10 (Fundamental Theorem of the Local Theory of Curves)**.** *For any smooth functions $k : [a,b] \to (0,\infty)$ and $\tau : [a,b] \to \mathbb{R}$, there exists a regular curve $\gamma : [a,b] \to \mathbb{R}^3$ parameterised by arc length such that $k_\gamma = k$ and $\tau_\gamma = \tau$. Moreover, γ is unique up to rigid motions of $\mathbb{R}^3$.*

Proof. Step 1: Uniqueness.
Suppose that regular curves $\gamma, \phi : [a,b] \to \mathbb{R}^3$ are parameterised by arc length and have curvature k and torsion τ. There exists a rigid motion such that

$$\begin{pmatrix} T_\gamma(a) \\ N_\gamma(a) \\ B_\gamma(a) \end{pmatrix} = A \begin{pmatrix} T_\phi(a) \\ N_\phi(a) \\ B_\phi(a) \end{pmatrix}.$$

Using Lemma 2.4.8 we can apply this rigid motion onto the curve ϕ whilst maintaining the curvature and torsion values. Henceforth, γ and ϕ will refer to the curves such that $T_\gamma(a) = T_\phi(a)$, $N_\gamma(a) = N_\phi(a)$ and $B_\gamma(a) = B_\phi(a)$.

Observe that,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle T_\phi - T_\gamma, T_\phi - T_\gamma\rangle + \langle N_\phi - N_\gamma, N_\phi - N_\gamma\rangle + \langle B_\phi - B_\gamma, B_\phi - B_\gamma\rangle\right) \\
&\quad = \left\langle T_\phi - T_\gamma, T'_\phi - T'_\gamma\right\rangle + \left\langle N_\phi - N_\gamma, N_\phi - N'_\gamma\right\rangle + \left\langle B_\phi - B_\gamma, B'_\phi - B'_\gamma\right\rangle \\
&\quad = k\langle T_\phi - T_\gamma, N_\phi - N_\gamma\rangle + (\tau\langle N_\phi - N_\gamma, B_\phi - B_\gamma\rangle - k\langle N_\phi - N_\gamma, T_\phi - T_\gamma\rangle) - \tau\langle B_\phi - B_\psi, N_\phi - N_\psi\rangle \\
&\quad = 0.
\end{aligned}$$

Therefore, the expression $\langle T_\phi - T_\gamma, T_\phi - T_\gamma\rangle + \langle N_\phi - N_\gamma, N_\phi - N_\gamma\rangle + \langle B_\phi - B_\gamma, B_\phi - B_\gamma\rangle$ is constant. At $t = a$ this expression is equal to zero, and so we deduce that

$$\langle T_\phi - T_\gamma, T_\phi - T_\gamma\rangle + \langle N_\phi - N_\gamma, N_\phi - N_\gamma\rangle + \langle B_\phi - B_\gamma, B_\phi - B_\gamma\rangle \equiv 0.$$

Consequently, we must have $T_\phi - T_\gamma \equiv 0$, $N_\phi - N_\gamma \equiv 0$ and $B_\phi - B_\psi \equiv 0$, thus

$$\begin{aligned}
\phi(t) &= \phi(a) + \int_a^t T_\phi(s)\,\mathrm{d}s \\
&= \phi(a) + \int_a^t T_\gamma(s)\,\mathrm{d}s \\
&= \phi(a) - \gamma(a) + \gamma(t).
\end{aligned}$$

Therefore, $\phi \equiv \gamma + \mathbf{c}$ for $\mathbf{c} = \phi(a) - \gamma(a) \in \mathbb{R}^3$ a constant. Hence, γ is unique up to rigid motions.

Step 2: Existence.

For $k : [a,b] \to (0,\infty)$ and $\tau : [a,b] \to \mathbb{R}$, fix (T_a, N_a, B_a) a positively oriented orthonormal basis of $\mathbb{R}^3$, and consider the unique functions $T, N, B : [a,b] \to \mathbb{R}^3$, given by Proposition 2.4.9, such that

$$\begin{pmatrix} T'(t) \\ N'(t) \\ B'(t) \end{pmatrix} = \begin{pmatrix} 0 & k(t) & 0 \\ -k(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix} \tag{2.4.1}$$

with

$$\begin{pmatrix} T(a) \\ N(a) \\ B(a) \end{pmatrix} = \begin{pmatrix} T_a \\ N_a \\ B_a \end{pmatrix}.$$

Let

$$M(t) := \begin{pmatrix} - & T(t) & - \\ - & N(t) & - \\ - & B(t) & - \end{pmatrix}.$$

Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(M^\top(t)M(t)\right) &= (M'(t))^\top M(t) + M(t)^\top M'(t) \\
&= \begin{pmatrix} 0 & k(t) & 0 \\ -k(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} M(t)^\top M(t) + M(t)^\top M(t) \begin{pmatrix} 0 & -k(t) & 0 \\ k(t) & 0 & -\tau(t) \\ 0 & \tau(t) & 0 \end{pmatrix}.
\end{aligned}$$

Using the orthonormality of (T_a, N_a, B_a) it follows that $M^\top(a)M(a) = I$. Therefore, $M^\top(t)M(t)$ solves the linear system

$$A'(t) = \begin{pmatrix} 0 & k(t) & 0 \\ -k(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} A(t) + A(t) \begin{pmatrix} 0 & -k(t) & 0 \\ k(t) & 0 & -\tau(t) \\ 0 & \tau(t) & 0 \end{pmatrix}$$

with $A(a) = I$. Another solution to which is $A \equiv I$, therefore, by Proposition 2.4.9 it follows that $M(t)^\top M(t) = I$ for all $t \in [a,b]$. In other words, $(T(t), N(t), B(t))$ is an orthonormal basis. Let $\gamma : [a,b] \to \mathbb{R}^3$ be given by

$$\gamma(t) = \int_a^t T(s)\,\mathrm{d}s$$

for $t \in [a, b]$. Let T_γ, N_γ and B_γ be the tangent, unit normal, and binormal vector of γ. Clearly, $T_\gamma = T$ which means that $k_\gamma \equiv k$. Then,

$$\begin{aligned} N_\gamma(t) &= \frac{T'_\gamma(t)}{|T'_\gamma(t)|} \\ &\overset{(2.4.1)}{=} \frac{T'(t)}{|T'(t)|} \\ &= \frac{k(t)N(t)}{|k(t)N(t)|} \\ &= \frac{k(t)N(t)}{k(t)} \\ &= N(t), \end{aligned}$$

where we have used the fact that $|k(t)| = k(t)$ and $|N(t)| = 1$. Thus,

$$\begin{aligned} B_\gamma(t) &= T_\gamma(t) \times N_\gamma(t) \\ &= T(t) \times N(t) \\ &= B(t) \end{aligned}$$

and so

$$\begin{aligned} -\tau_\gamma(t)N(t) &= -\tau_\gamma(t)N_\gamma(t) \\ &= B'_\gamma(t) \\ &= B'(t) \\ &\overset{(2.4.1)}{=} -\tau(t)N(t) \end{aligned}$$

which implies that $\tau_\gamma \equiv \tau$. ☐

Remark 2.4.11. *Note how the range of k in Theorem 2.4.10 does not include zero such that torsion for the corresponding curve is well-defined.*

Corollary 2.4.12. *If $\gamma([a, b])$ is a regular curve in $\mathbb{R}^3$ that has torsion $\tau \equiv 0$, and curvature $k(t) \equiv c \in \mathbb{R}_{>0}$, then $\gamma([a, b])$ is a segment of a circle of radius $\frac{1}{c}$.*

Proof. By generalising the argument of statement 1 from Example 2.4.6, it follows that for $r > 0$ the curve of a circle of radius r given by $\gamma_r : [0, 2\pi) \to \mathbb{R}^3$ where

$$\gamma_r(t) = \left(r\cos\left(\frac{t}{r}\right), r\sin\left(\frac{t}{r}\right), 0 \right)$$

has curvature $k_r \equiv \frac{1}{r}$ and torsion $\tau_r \equiv 0$. Therefore, if a curve $\phi : [a, b] \to \mathbb{R}^3$ is such that $k_\phi \equiv c \in \mathbb{R}$ and $\tau_\phi \equiv 0$, it follows by Theorem 2.4.10 that under rigid motions in $\mathbb{R}^3$ the curve ϕ is equivalent to $\gamma_{\frac{1}{c}}$. Thus, ϕ is a segment of a circle of radius $\frac{1}{c}$. ☐

Exercise 2.4.13. *Let $I \subseteq \mathbb{R}$ be an interval and let $\gamma : I \to \mathbb{R}^3$ be a regular curve parameterised by arc length such that $\gamma''(t) \neq 0$, $k'(t) \neq 0$ and $\tau(t) \neq 0$ for all $t \in I$, where k is the curvature and τ is the torsion of γ. Then $\gamma(I)$ lies on a sphere of radius $R > 0$ if and only if*

$$\frac{1}{k(t)^2} + \frac{k'(t)^2}{k(t)^4\tau(t)^2} = R^2$$

for all $t \in I$.

2.5 Curve in $\mathbb{R}^2$

Theorem 2.4.10 can be considered a local theorem as its conclusions and derivation can be restricted to a neighbourhood of the curve, and it does not take into account the global structure of the curve. To obtain a global result for curves we focus on planar curves.

2.5.1 Signed Curvature

Consider $\gamma : [a, b] \to \mathbb{R}^2$ a regular curve, not necessarily parameterised by arc length, given by

$$\gamma(t) = (x(t), y(t))$$

such that

$$\gamma'(t) = (x'(t), y'(t)).$$

Then let $n : [a, b] \to \mathbb{R}$ be given by

$$n(t) = \frac{(-y'(t), x'(t))}{|\gamma'(t)|}. \tag{2.5.1}$$

One can interpret the construction of n as rotating the tangent vector to γ at t and then normalising. Thus, n is a normal tangent vector to γ at t. Consequently, $\left\{\frac{\gamma'(t)}{|\gamma'(t)|}, n(t)\right\}$ is a positively oriented orthonormal basis for $\mathbb{R}^2$.

Definition 2.5.1. *Suppose $\gamma : [a, b] \to \mathbb{R}^2$ is a regular curve parameterised by arc length. The signed curvature of γ is $\kappa : [a, b] \to \mathbb{R}$ given by*

$$\kappa(t) = \langle n(t), \gamma''(t)\rangle = x'(t)y''(t) - y'(t)x''(t). \tag{2.5.2}$$

Remark 2.5.2. *Recall, that for $\gamma : [a, b] \to \mathbb{R}^2$ a regular curve parameterised by arc length, its curvature is given by*

$$k(t) = |\gamma''(t)|\,.$$

Thus,

$$\begin{aligned} k(t) &= |\gamma''(t)| \\ &\overset{(1)}{=} |\langle \gamma'(t), \gamma''(t)\rangle| + |\langle n(t), \gamma''(t)\rangle| \\ &\overset{(2)}{=} |\langle n(t), \gamma''(t)\rangle| \end{aligned}$$

where in (1) we are expanding $|\gamma''(t)|$ in terms of the orthonormal basis $\{\gamma'(t), n(t)\}$, in (2) we use the fact that $\gamma'(t)$ and $\gamma''(t)$ are orthogonal. Consequently, we observe that $|\kappa(t)| = k(t)$ for all $t \in [a, b]$, which means that κ changes sign when the curve is traversed in the opposite direction.

Example 2.5.3. *Consider $\gamma_+ : [0, 2\pi) \to \mathbb{R}^2$ given by*

$$t \mapsto (\cos(t), \sin(t)),$$

and $\gamma_- : [0, 2\pi) \to \mathbb{R}^2$ given by

$$t \mapsto (\cos(t), -\sin(t)).$$

Then the normal vector to γ_+ is

$$n_+(t) = (-\cos(t), -\sin(t))$$

and so the signed curvature of γ_+ is

$$\begin{aligned}\kappa_+(t) &= \left\langle \begin{pmatrix} -\cos(t) \\ -\sin(t) \end{pmatrix}, \begin{pmatrix} -\cos(t) \\ -\sin(t) \end{pmatrix} \right\rangle \\ &= \cos^2(t) + \sin^2(t) \\ &= 1.\end{aligned}$$

Similarly, the normal vector to γ_- is

$$n_-(t) = (\cos(t), -\sin(t)),$$

so that the signed curvature of γ_- is

$$\kappa_-(t) = \left\langle \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}, \begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix} \right\rangle = -1$$

This verifies Remark 2.5.2 which states that signed curvature changes sign when the curve is traversed in the opposite direction.

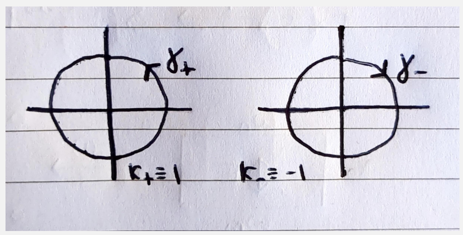


Figure 2.5.1: Curves traversed in different directions have opposite signs for their signed curvature.

Proposition 2.5.4. *Let $\gamma : [a, b] \to \mathbb{R}^2$ be a regular curve. Then*

$$\kappa(t) = \frac{\langle \gamma''(t), n(t) \rangle}{|\gamma'(t)|^2}.$$

Proof. Let $\psi(s) = (\gamma \circ f)(s)$ be a reparameterisation of γ with $f' > 0$ such that $|\psi'| \equiv 1$. Let

$$\psi(s) = (x(f(s)), y(f(s))).$$

From Proposition 2.2.6 we have $f = h^{-1}$ where

$$h(t) = \int_a^t |\gamma'(t)| \,\mathrm{d}t.$$

Moreover, using (2.2.1) and (2.2.2) we note that

$$f'(s) = \frac{1}{|\gamma'(f(s))|}.$$

Since $\gamma(f(s)) = \psi(s)$ with γ and ψ parameterised in the same direction, it follows that $n_\gamma(f(s)) = n_\psi(s)$. Therefore,

$$\begin{aligned}n_\gamma(f(s)) &= n_\psi(s) \\ &= \frac{(-(y(f(s)))', (x(f(s)))')}{|\psi'(s)|} \\ &= (-(y(f(s)))', (x(f(s)))') \\ &= f'(s)\left(-y'(f(s)), x'(f(s))\right).\end{aligned}$$

On the other hand, $k_\gamma(f(s)) = k_\psi(s)$ and so

$$|\kappa_\psi(s)| = k_\psi(s) = k_\gamma(f(s)) = |\kappa_\gamma(f(s))|.$$

Since γ and ψ are oriented in the same direction we have that $\kappa_\psi(s) = \kappa_\gamma(f(s))$. Therefore, using (2.5.2) it follows that

$$\begin{aligned}
\kappa_\gamma(f(s)) &= \kappa_\psi(s) \\
&= (x(f(s)))'(y(f(s)))'' - (y(f(s)))'(x(f(s)))'' \\
&= (x'(f(s))y''(f(s)) - y'(f(s))x''(f(s)))\,(f'(s))^3 \\
&= \langle \gamma''(f(s)), (-y'(f(s)), x'(f(s)))\rangle\,(f'(s))^3 \\
&= \frac{\langle \gamma''(f(s)), n_\gamma(f(s))\rangle}{|\gamma'(f(s))|^2} \\
&\overset{t=f(s)}{=} \frac{\langle \gamma''(t), n_\gamma(t)\rangle}{|\gamma'(t)|^2}.
\end{aligned}$$

$\square$

Example 2.5.5.

1. *Consider the ellipse $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by*

$$\gamma(t) = (a\cos(t), b\sin(t)).$$

Then

$$\gamma'(t) = (-a\sin(t), b\cos(t)),$$

and

$$\gamma''(t) = (-a\cos(t), -b\sin(t)).$$

Hence,

$$|\gamma'(t)| = \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}$$

and

$$n_\gamma(t) = \frac{(-b\cos(t), -a\sin(t))}{\sqrt{a^2\sin^2(t) + b^2\cos^2(t)}}.$$

Therefore,

$$\kappa_\gamma(t) = \frac{ab}{\left(a^2\sin^2(t) + b^2\cos^2(t)\right)^{\frac{3}{2}}}.$$

2. *Let $\gamma : [a, b] \to \mathbb{R}^2$ be the graph of $f : [a, b] \to \mathbb{R}$. That is,*

$$\gamma(t) = (t, f(t)).$$

Then

$$\gamma'(t) = (1, f'(t))$$

and

$$\gamma''(t) = (0, f''(t)).$$

Hence,

$$|\gamma'(t)| = \sqrt{1 + (f'(t))^2},$$

and

$$n_\gamma(t) = \frac{(-f'(t), 1)}{\sqrt{1 + (f'(t))^2}}.$$

Therefore,

$$\kappa_\gamma(t) = \frac{f''(t)}{\left(1 + (f'(t))^2\right)^{\frac{3}{2}}}.$$

2.5.2 Winding Number

Definition 2.5.6. *A curve $\gamma : [a, b] \to \mathbb{R}^2$ is a smooth closed curve if $\gamma^{(k)}(a) = \gamma^{(k)}(b)$ for every $k \in \mathbb{N}$.*

Remark 2.5.7. *If γ is a smooth closed curve, then the map $T(t) = \gamma'(t) : [a, b] \to \mathbb{R}^2$ is also a smooth closed curve.*

Observe how a curve in $\mathbb{R}^2$ can be associated with a curve in $\mathbb{C}$. In particular, for $\gamma : [a, b] \to \mathbb{R}^2$ given by

$$\gamma(t) = (x(t), y(t)),$$

one can consider $\gamma : [a, b] \to \mathbb{C}$ where

$$\gamma(t) = x(t) + iy(t).$$

Definition 2.5.8. *Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve. Then*

$$\omega(\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z} \,\mathrm{d}z$$

is referred to as the winding number of γ.

Remark 2.5.9. *The winding number $\omega(\gamma)$ measures the number of times γ winds around the origin. In particular, a positive winding number corresponds to a counter-clockwise traversal around the origin, whereas a negative winding number corresponds to a clockwise traversal around the origin.*

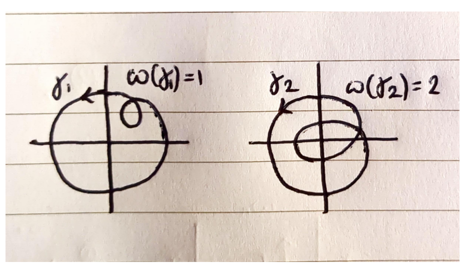


Figure 2.5.2: Winding number measures the number of times a curve winds around the origin.

Proposition 2.5.10. *Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth closed curve given by*

$$\gamma(t) = x(t) + iy(t).$$

Then

$$\omega(\gamma) = \frac{1}{2\pi} \int_a^b \frac{\langle \gamma'(t), -y(t) + x(t) \rangle}{|\gamma(t)|^2} \, \mathrm{d}t.$$

Proof. Observe that

$$\begin{aligned}
\omega(\gamma) &= \frac{1}{2\pi i} \int_\gamma \frac{1}{z} \, \mathrm{d}z \\
&\overset{z=\gamma(t)}{=} \frac{1}{2\pi i} \int_a^b \frac{x'(t) + iy'(t)}{x(t) + iy(t)} \, \mathrm{d}t \\
&= \frac{1}{2\pi i} \int_a^b \frac{(x'(t) + iy'(t))\,(x(t) - iy(t))}{x(t)^2 + y(t)^2} \, \mathrm{d}t \\
&= \frac{1}{2\pi i} \int_a^b \frac{x(t)x'(t) + y(t)y'(t)}{x(t)^2 + y(t)^2} + i \frac{x(t)y'(t) - y(t)x'(t)}{x(t)^2 + y(t)^2} \, \mathrm{d}t \\
&= \frac{1}{2\pi i} \left(\left[\frac{1}{2} \log\left(x(t)^2 + y(t)^2\right) \right]_a^b + i \int_a^b \frac{x(t)y'(t) - y(t)x'(t)}{x(t)^2 + y(t)^2} \, \mathrm{d}t \right) \\
&\overset{(1)}{=} \frac{1}{2\pi} \int_a^b \frac{\langle \gamma'(t), (-y(t), x(t)) \rangle}{|\gamma(t)|^2} \, \mathrm{d}t
\end{aligned}$$

where in (1) we have used the fact that γ is closed. $\square$

Theorem 2.5.11. *If $\gamma : [a,b] \to \mathbb{R}^2$ is a regular closed curve, given by*

$$\gamma(t) = (x(t), y(t)),$$

then

$$\omega(\gamma') = \frac{1}{2\pi} \int_a^b \kappa(t) \, \mathrm{d}t,$$

where κ is the signed curvature of γ.

Proof. Using Remark 2.5.7, Proposition 2.5.10 can be applied to $\gamma' : [a,b] \to \mathbb{C}$ to give

$$\begin{aligned}
\omega(\gamma') &= \frac{1}{2\pi} \int_a^b \frac{\langle \gamma''(t), (-y'(t), x'(t)) \rangle}{|\gamma'(t)|^2} \, \mathrm{d}t \\
&\overset{\text{Prop 2.5.4}}{=} \frac{1}{2\pi} \int_a^b \kappa(t) \, \mathrm{d}t.
\end{aligned}$$

$\square$

Remark 2.5.12.

1. *The quantity $\omega(\gamma')$ for $\gamma : [a,b] \to \mathbb{C}$ a regular closed curve is referred to as its index, or turning number and is denoted $\operatorname{Ind}(\gamma)$. The index of a curve measures the number of times the unit tangent vector rotates around the origin as ϕ is traversed, consequently, we have that $\operatorname{Ind}(\gamma) \in \mathbb{Z}$.*

2. *Theorem 2.5.11 concerns global properties of a curve. In particular, it relates a topological property, the turning number, to a geometrical property, the curvature.*

3. *One can observe that the turning number of a curve is robust to perturbations of γ. Geometrically, a small perturbation is unlikely to fully rotate the tangent vector of the curve. Concretely, the sign of the curvature means that small perturbations are likely to be cancelled out as they are integrated over.*

Exercise 2.5.13. *For each $n \in \mathbb{Z}$ find a regular closed curve $\gamma : I \to \mathbb{R}^2$, where $I \subseteq \mathbb{R}$ is an appropriate interval, such that $\operatorname{Ind}(\gamma) = n$.*

2.6 Solutions to Exercises

Exercise 2.3.8

Solution.

1. The tangent vector to a point on a circle is perpendicular to the line from the point of the circle to the centre of the circle. Therefore, the circle C_v is tangent to γ at $\gamma(t_0)$ if and only if $\langle \gamma'(t_0), \gamma(t_0) - v \rangle = 0$. As
$$f_v'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma(t) - v, \gamma(t) - v\rangle = 2\left\langle \gamma'(t), \gamma(t) - v\right\rangle,$$
it follows that C_v is tangent to γ at $\gamma(t_0)$ if and only if $f_v'(t_0) = 0$.

2. Let $v = \gamma(t_0) + \frac{\gamma''(t_0)}{|\gamma''(t_0)|^2}$. Then
$$f_v'(t_0) = -2\left\langle \gamma'(t_0), \frac{\gamma''(t_0)}{|\gamma''(t_0)|^2}\right\rangle = 0,$$
as $\gamma'(t_0)$ and $\gamma''(t_0)$ are perpendicular to each other, Proposition 2.3.6. Note that
$$f_v''(t_0) = 2\left\langle \gamma''(t_0), \gamma(t_0) - v\right\rangle + 2\left\langle \gamma'(t_0), \gamma'(t_0)\right\rangle = 2\left\langle \gamma''(t_0), \gamma(t_0) - v\right\rangle + 2$$
as γ is parameterised by arc length. Therefore,
$$f_v''(t_0) = -2\left\langle \gamma''(t_0), \frac{\gamma''(t_0)}{|\gamma''(t_0)|^2}\right\rangle + 2 = -2 + 2 = 0.$$
Moreover, the radius of C_v is given by
$$|v - \gamma(t_0)| = \frac{|\gamma''(t_0)|}{|\gamma''(t_0)|^2} = \frac{1}{|\gamma''(t_0)|} = \frac{1}{k_\gamma(t_0)}.$$

□

Exercise 2.4.13

Solution. ($\Rightarrow$). Suppose that $\gamma(I)$ lies on the circle of radius R centred at p_0. Namely, $\langle \gamma(t) - p_0, \gamma(t) - p_0\rangle = R^2$ for all $t \in I$. Differentiating this it follows that
$$\langle \gamma'(t), \gamma(t) - p_0\rangle = 0,$$
which when differentiated gives
$$\langle \gamma''(t), \gamma(t) - p_0\rangle + \langle \gamma'(t), \gamma'(t)\rangle = 0$$
for all $t \in I$. In particular,
$$k(t)\langle N(t), \gamma(t) - p_0\rangle = -1,$$
where we have used the fact that γ is parameterised by arc length. Differentiating this expression we deduce that
$$k'(t)\langle N(t), \gamma(t) - p_0\rangle + k(t)\left\langle N'(t), \gamma(t) - p_0\right\rangle + k(t)\left\langle N(t), \gamma'(t)\right\rangle = 0.$$
Thus, since $N'(t) = \tau(t)B(t) - k(t)T(t)$ it follows that
$$\frac{k'(t)}{k(t)} + k(t)\left\langle \tau(t)B(t) - k(t)T(t), \gamma(t) - p_0\right\rangle = 0.$$

Or equivalently,

$$\langle B(t), \gamma(t) - p_0 \rangle = -\frac{k'(t)}{k(t)^2\tau(t)}.$$

Therefore, since $\{T(t), N(t), B(t)\}$ is an orthonormal basis, we have

$$\begin{aligned}
R^2 &= \langle \gamma(t) - p_0, \gamma(t) - p_0 \rangle \\
&= \langle T(t), \gamma(t) - p_0 \rangle^2 + \langle N(t), \gamma(t) - p_0 \rangle^2 + \langle B(t), \gamma(t) - p_0 \rangle^2 \\
&= 0 + \left(-\frac{1}{k(t)}\right)^2 + \left(-\frac{k'(t)}{k(t)^2\tau(t)}\right)^2 \\
&= \frac{1}{k(t)^2} + \frac{k'(t)^2}{k(t)^4\tau(t)^2}.
\end{aligned}$$

($\Leftarrow$). Let $\rho = \frac{1}{k}$ and $\sigma = \frac{1}{\tau}$ so that by assumption we have that

$$\rho^2 + (\rho'\sigma)^2 = R^2. \tag{2.6.1}$$

Differentiating this it follows that

$$\rho\rho' + (\rho'\sigma)(\rho'\sigma)' = 0,$$

thus

$$\frac{\rho}{\sigma} + (\rho'\sigma)' = 0. \tag{2.6.2}$$

Now let $\alpha : I \to \mathbb{R}^3$ be given by

$$\alpha(t) = \gamma(t) + \rho(t)N(t) + \rho'(t)\sigma(t)B(t),$$

where $\{T(t), N(t), B(t)\}$ is the Frenet frame of γ at $\gamma(t)$. It follows that

$$\begin{aligned}
\alpha'(t) &= T(t) + \rho'(t)N(t) + \rho(t)N'(t) + (\rho'(t)\sigma(t))' B(t) + (\rho'(t)\sigma(t)) B'(t) \\
&= T(t) + \rho'(t)N(t) + \rho(t)\left(\frac{B(t)}{\sigma(t)} - \frac{T(t)}{\rho(t)}\right) + (\rho'(t)\sigma(t))' B(t) + (\rho'(t)\sigma(t))\left(-\frac{N(t)}{\sigma(t)}\right) \\
&= B(t)\left(\frac{\rho(t)}{\sigma(t)} + (\rho'(t)\sigma(t))'\right) \\
&\overset{(2.6.2)}{=} 0.
\end{aligned}$$

Therefore, $\alpha(t) = p_0$ for some $p_0 \in \mathbb{R}^3$ and every $t \in I$. In particular, as $\{T(t), N(t), B(t)\}$ is an orthonormal basis it follows that

$$|\gamma(t) - p_0|^2 = |\rho(t)N(t) + \rho'(t)\sigma(t)B(t)|^2 = \rho(t)^2 + (\rho'(t)\sigma(t))^2 \overset{(2.6.1)}{=} R^2.$$

That is, $\gamma(t)$ lies on the sphere of radius R centred at p_0. □

Exercise 2.5.13

Solution. For $n > 0$ let $\gamma_n : [0, 2\pi n) \to \mathbb{C}$ be given by

$$\gamma_n(t) = \cos(t) + i\sin(t).$$

Then

$$\gamma_n'(t) = -\sin(t) + i\cos(t),$$

and so γ_n is parameterised by arc length. Moreover,

$$\gamma_n''(t) = -\cos(t) - i\sin(t),$$

and so

$$\kappa_{\gamma_n}(t) = (-\sin(t))(-\sin(t)) - (-\cos(t))(\cos(t)) = 1.$$

Therefore,

$$\mathrm{Ind}(\gamma_n) = \frac{1}{2\pi}\int_0^{2\pi n} \kappa_{\gamma_n}(t)\,\mathrm{d}t = \frac{1}{2\pi}(2\pi n) = n.$$

For $n < 0$ let $\gamma_n : [0, 2\pi|n|) \to \mathbb{C}$ be given by

$$\gamma_n(t) = \cos(t) - i\sin(t).$$

Then

$$\gamma_n'(t) = -\sin(t) - i\cos(t)$$

and so γ_n is parameterised by arc length. Moreover,

$$\gamma_n''(t) = -\cos(t) + i\sin(t)$$

and so

$$\kappa_{\gamma_n}(t) = (-\sin(t))(\sin(t)) - (-\cos(t))(-\cos(t)) = -1.$$

Therefore,

$$\mathrm{Ind}(\gamma_n) = \frac{1}{2\pi}\int_0^{2\pi|n|} (-1)\,\mathrm{d}t = \frac{1}{2\pi}\,(-2\pi|n|) = n.$$

For $n = 0$ consider $\gamma_0 : [0, 4\pi]$ given by

$$\gamma_0(t) = \begin{cases} -\cos(t) + 1 + i\sin(t) & t \in [0, 2\pi] \\ \cos(t) - 1 - i\sin(t) & t \in [2\pi, 4\pi]. \end{cases}$$

Then

$$\gamma_0'(t) = \begin{cases} \sin(t) + i\cos(t) & t \in [0, 2\pi] \\ -\sin(t) - i\cos(t) & t \in [2\pi, 4\pi], \end{cases}$$

and so γ_0 is parameterised by arc length. Moreover,

$$\gamma_0''(t) = \begin{cases} \cos(t) - i\sin(t) & t \in [0, 2\pi] \\ -\cos(t) - i\sin(t) & t \in [2\pi, 4\pi], \end{cases}$$

and so

$$\kappa_{\gamma_0}(t) = \begin{cases} (\sin(t))(-\sin(t)) - (\cos(t))(\cos(t)) & t \in [0, 2\pi] \\ (-\sin(t))(-\sin(t)) - (-\cos(t))(-\cos(t)) & t \in [2\pi, 4\pi] \end{cases} = \begin{cases} -1 & t \in [0, 2\pi] \\ 1 & t \in [2\pi, 4\pi]. \end{cases}$$

Therefore,

$$\mathrm{Ind}(\gamma_0) = \int_0^{2\pi} -1\,\mathrm{d}t + \int_{2\pi}^{4\pi} 1\,\mathrm{d}t = 0.$$

$\square$

3 Surfaces

Informally, a surface in $\mathbb{R}^3$ is a subset $S \subseteq \mathbb{R}^3$ that locally looks like $\mathbb{R}^2$. As many notions for curves involve taking derivatives, it is essential that on surfaces we ought to be able to perform classical calculus. Recall that for curves a regularity condition ensured that one could perform calculus along the curve. Namely, a curve $\gamma : [a, b] \to \mathbb{R}^n$ is said to be regular if $|\gamma'(0)| \neq 0$ for all $t \in [a, b]$. The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = \left(t^3, t^2\right)$ from statement 6 of Example 2.1.3 is not regular, as it has a cusp at the origin which means that the derivative at the origin is not well-defined. Consequently, the tangent vector field to γ is not well-defined and at the origin it is not locally homeomorphic to a line. We would similarly like to define regular surfaces to avoid such issues. For a curve γ the regularity condition $|\gamma'(t)| \neq 0$ can be equivalently stated as $\gamma'(t)$ is injective for all $t \in [a, b]$. This rephrasing is used to generalise regular curves to regular surfaces.

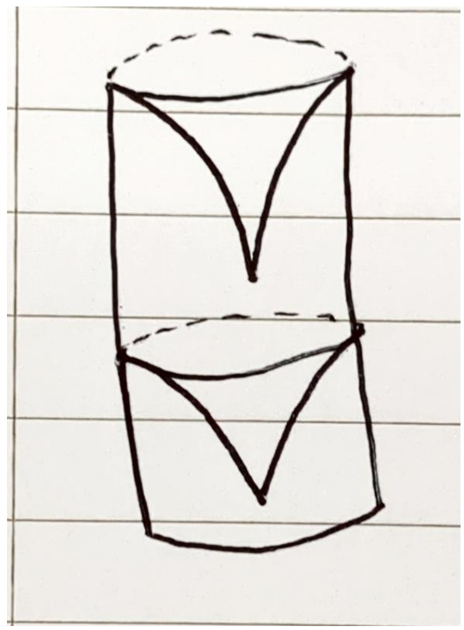


Figure 3.0.1: An example of a surface with a cusp that would not facilitate calculus to be performed along the surface.

3.1 Charts

Definition 3.1.1. *A regular surface is a set $S \subseteq \mathbb{R}^3$ such that for all $p \in S$ there exists an open neighbourhood $V \subseteq \mathbb{R}^3$ of p, an open set $U \subseteq \mathbb{R}^2$, and a smooth map $\phi : U \to \mathbb{R}^3$ such that*

1. *$\phi(U) = V \cap S$,*
2. *$\phi : U \to V \cap S$ is a homeomorphism, and*
3. *for all $q \in U$ the derivative of ϕ at q denoted $\mathrm{d}\phi_q : \mathbb{R}^2 \to \mathbb{R}^3$, is injective.*

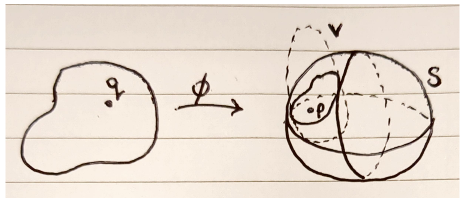


Figure 3.1.1: A regular surface locally resembles a plane in $\mathbb{R}^2$, through the use of a chart ϕ.

Remark 3.1.2.

1. *The pair (ϕ, U) from Definition 3.1.1 is referred to as a chart, or local parameterisation, of S at p.*
2. *Statements 1 and 2 of Definition 3.1.1 say that S locally resembles $\mathbb{R}^2$. Statement 3 of Definition 3.1.1 ensures S is sufficiently regular near p, it is the generalisation of the regularity condition for curves.*

Let S be a regular surface with (ϕ, U) a chart for S at $p \in S$. Suppose that for $(u, v) \in U$ we have

$$\phi(u, v) = (x(u, v), y(u, v), z(u, v)).$$

Then by statement 3 of Definition 3.1.1 we have that

$$\mathrm{d}\phi_q = \begin{pmatrix} \frac{\partial x}{\partial u}(q) & \frac{\partial x}{\partial v}(q) \\ \frac{\partial y}{\partial u}(q) & \frac{\partial y}{\partial v}(q) \\ \frac{\partial z}{\partial u}(q) & \frac{\partial z}{\partial v}(q) \end{pmatrix}$$

is injective. Observe that $\mathrm{d}\phi_q$ is injective if and only if the columns $\frac{\partial \phi}{\partial u}(q)$ and $\frac{\partial \phi}{\partial v}(q)$ are linearly independent. Note that for $q = (u_0, v_0)$ the curves

$$u \mapsto \phi(u, v_0)$$

and

$$v \mapsto \phi(u_0, v)$$

are contained in S with the tangent vectors at q given by $\partial_u \phi(u_0, v_0)$ and $\partial_v \phi(u_0, v_0)$ respectively. Therefore, $\mathrm{d}\phi_q$ is injective if and only if the tangent vectors to the curves span a plane.

Example 3.1.3.

1. *The set $\{z = 0\} \subseteq \mathbb{R}^3$ is a regular surface. It is specified by the single chart $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by $(u, v) \mapsto (u, v, 0)$. Observe that ϕ is homeomorphic to its image and*

$$\mathrm{d}\phi_q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is injective for all $q \in \mathbb{R}^2$. Note that $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\phi(u, v) = \left(u^3, v^3, 0\right)$ would not be a chart for $\{z = 0\}$ as its differential is zero at the origin.

2. *Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be smooth, then*

$$\Gamma_f := \left\{(x, y, z) \in \mathbb{R}^3 : f(x, y) = z\right\}$$

is a regular surface. It is specified by the single chart

$$\phi(u, v) = (u, v, f(u, v))$$

as ϕ is homeomorphic onto its image and

$$\mathrm{d}\phi_q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_u f & \partial_v f \end{pmatrix}$$

is injective for all $q \in \mathbb{R}^2$.

3. *The unit sphere $\mathrm{S}^2 = \left\{(x,y,z) \in \mathbb{R}^3 : |x| = 1\right\}$ is a regular surface. However, more than one chart is required to specify S^2 as S^2 is not homeomorphic to $\mathbb{R}^2$. Let $\phi_N : \mathbb{R}^2 \to \mathrm{S}^2 \setminus (0,0,1)$ and $\phi_S : \mathbb{R}^2 \to \mathrm{S}^2 \setminus (0,0,-1)$ be the stereographic projections from the north of S^2, that is $(0,0,1)$, and the south of S^2, that is $(0,0,-1)$, respectively. More specifically,*

$$\phi_N(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2}\right)$$

and

$$\phi_S(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-\left(u^2+v^2\right)}{1+u^2+v^2}\right).$$

Sets that are not regular surfaces include the following.

4. *The cone*

$$\left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = z^2, z \geq 0\right\}$$

is not a regular surface. There is no smooth chart at the origin.

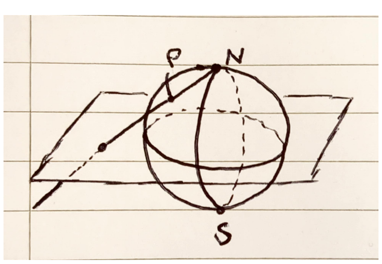
5. *Let $\phi : (0,2\pi) \times (-1,1) \to \mathbb{R}^3$ be given by*

$$\phi(u,v) = (\sin(u), \sin(2u), v)\,.$$

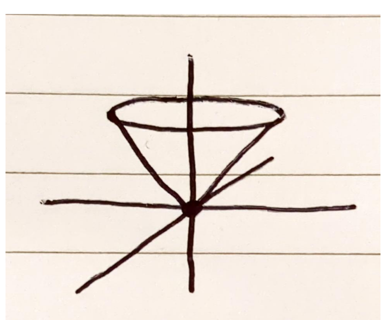
Note that ϕ is smooth, and it is also injective since it is defined on open intervals. Moreover,

$$\mathrm{d}\phi_q = \begin{pmatrix} \cos(u) & 0 \\ 2\cos(2u) & 0 \\ 0 & 1 \end{pmatrix}$$

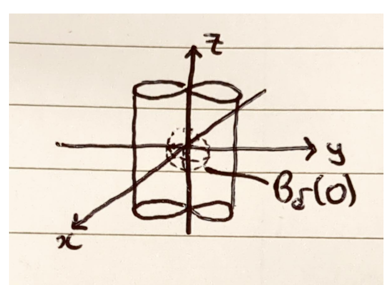
is injective for every $q = (u,v) \in U$. However, it is not homeomorphic to its image and thus is not a chart. More specifically, consider the point $(0,0,0) \in \operatorname{im}(\phi)$. Then the pre-image of the neighbourhood $V = B_\delta(0,0,0) \subseteq \mathbb{R}^3$ for any $\delta > 0$ will be of the form $(0,\epsilon_1) \cup (2\pi - \epsilon_2, 2\pi) \times (-\delta,\delta)$ which is disconnected and thus cannot be homeomorphic to V which is connected.



(a) The stereographic projection map.



(b) The cone, which is not a regular surface due to the point at the origin.



(c) A self-intersecting surface is not regular as charts are not homeomorphic to their images.

Figure 3.1.2: Illustrations supporting the examples and non-examples of surfaces given in Example 3.1.3.

3.2 Constructing Surfaces

Let $\gamma : I \to \mathbb{R}^2$ be a regular curve, with

$$\gamma(u) = (\lambda(u), \mu(u)).$$

Suppose that $\lambda(u) > 0$ for all $u \in I$. Then

$$S := \{(\lambda(u)\cos(v), \lambda(u)\sin(v), \mu(u)) : u \in I, v \in (0, 2\pi)\}$$

is a regular surface. More specifically, it is the surface of revolution of γ.

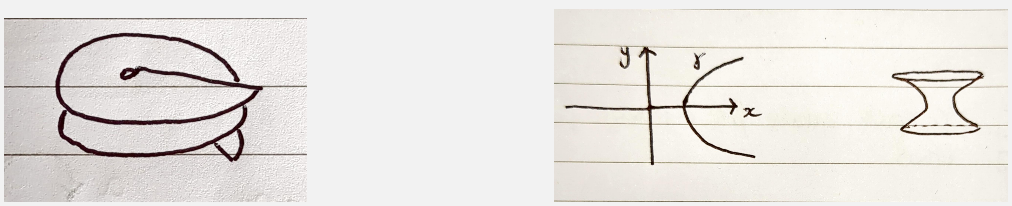
Example 3.2.1.

1. *The helicoid is the set of points*

$$\{(u\cos(v), u\sin(v), v) : u, v \in \mathbb{R}\}.$$

   A chart for the helicoid is given by $\phi(u, v) = (u\cos(v), u\sin(v), v)$.

2. *The catenoid is the surface of revolution of the catenary curve, namely* $\gamma : \mathbb{R} \to \mathbb{R}^2$ *where* $\gamma(u) = (\cosh(u), u)$.



(a) The helicoid.

(b) The catenoid.

Figure 3.2.1: Surfaces of revolution.

Surfaces may also be constructed as the level sets of functions.

Definition 3.2.2. *For* $n \geq 1$, *let* $U, V \subseteq \mathbb{R}^n$ *be open. Then a map* $f : U \to V$ *is a* $\mathcal{C}^k$*-diffeomorphism, for* $k \in \mathbb{N} \cup \{\infty\}$, *if* $f : U \to V$ *is a homeomorphism, with* $f : U \to V$ *and* $f^{-1} : V \to U$ *in* $\mathcal{C}^k$.

Remark 3.2.3. *Although Definition 3.2.2 distinguishes between diffeomorphisms of differing regularity, we will henceforth use diffeomorphism to refer to a* $\mathcal{C}^\infty$*-diffeomorphism.*

Example 3.2.4. *The function* $f(x) = x^3$ *is not a diffeomorphism. Although it is a homeomorphism, its inverse is not smooth.*

Theorem 3.2.5 (Inverse Function Theorem)**.** *Let* $\Omega \subseteq \mathbb{R}^n$ *be open,* $f : \Omega \to \mathbb{R}^n$ *a smooth map. Assume for some* $p \in \Omega$ *that* $\mathrm{d}f_p : \mathbb{R}^n \to \mathbb{R}^n$ *is invertible. Then, there is a neighbourhood* $U \subseteq \Omega$ *of* p *such that* $f : U \to f(U)$ *is a diffeomorphism.*

Proposition 3.2.6. *Let* $\Omega \subseteq \mathbb{R}^3$ *be open,* $F : \Omega \to \mathbb{R}$ *a smooth function, and* $c \in \mathbb{R}$. *Assume that* $\nabla F(p) \neq 0$ *for every* $p \in \Omega$. *Then*

$$S := F^{-1}(c) = \{(x, y, z) \in \Omega : F(x, y, z) = c\}$$

is a regular surface.

Proof. As $\nabla F(p) \neq 0$, at least one of $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ is non-zero at p. Without loss of generality suppose $\frac{\partial F}{\partial z}(p) \neq 0$ and consider $g : \Omega \to \mathbb{R}^3$ given by

$$g(x, y, z) = (x, y, F(x, y, z)).$$

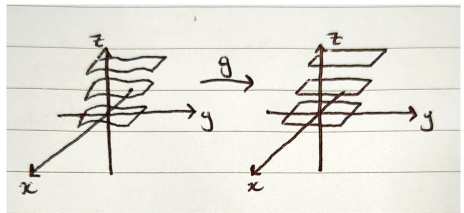


Figure 3.2.2

Then g is a smooth map on Ω with

$$\mathrm{d}g_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial F}{\partial x}(p) & \frac{\partial F}{\partial y}(p) & \frac{\partial F}{\partial z}(p) \end{pmatrix}.$$

Note that $\det(\mathrm{d}g_p) = \frac{\partial F}{\partial z}(p)$. Therefore, as by assumption we have $\frac{\partial F}{\partial z}(p) \neq 0$ it follows that $\mathrm{d}g_p$ is invertible and thus Theorem 3.2.5 can be applied. In particular, there exists a neighbourhood $U \subseteq \Omega$ of p such that $g : U \to g(U)$ is a diffeomorphism. Let

$$W := \left\{(u, v) \in \mathbb{R}^2 : (u, v, c) \in g^{-1}(U)\right\},$$

such that W is the projection of the intersection of the plane $z = c$ with $g^{-1}(U)$. Since $g^{-1}(U) \subseteq \mathbb{R}^3$ is open it follows that $W \subseteq \mathbb{R}^2$ is open. Let $\phi : W \to U$ be given by

$$\phi(u, v) = g^{-1}(u, v, c) = (u, v, h(u, v)),$$

where h is some smooth function. In particular, ϕ is a smooth homeomorphism onto its image, as Theorem 3.2.5 ensures g^{-1} is a smooth homeomorphism. Moreover,

$$\mathrm{d}\phi_q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_u h & \partial_v h \end{pmatrix}$$

is injective for all $q \in W$. Thus ϕ is a chart for $S = F^{-1}(c)$ at p. □

Definition 3.2.7. *A surface $S \subseteq \mathbb{R}^3$ is a regular level set if there is an open set $\Omega \subseteq \mathbb{R}^3$ and a smooth function $F : \Omega \to \mathbb{R}$ such that $S = F^{-1}(c)$ for some $c \in \mathbb{R}$ and $\nabla F(p) \neq 0$ for all $p \in S$.*

Example 3.2.8.

1. *Consider $\mathrm{S}^2 = \left\{x^2 + y^2 + z^2 = 1\right\}$. Let $\Omega = \mathbb{R}^3$, $F(x, y, z) = x^2 + y^2 + z^2$ and $c = 1$. Then $\mathrm{S}^2 = F^{-1}(c)$ and*

$$\nabla F(x, y, z) = (2x, 2y, 2z) \neq 0$$

for all $x, y, z \in \mathrm{S}^2$. Hence, using Proposition 3.2.6, it follows that S^2 is a surface. Indeed, it is the surface of the unit sphere in $\mathbb{R}^3$.

2. *Consider the torus*

$$S = \left\{(x, y, z) : \left(\sqrt{x^2 + y^2} - 2\right)^2 - z^2 = 1\right\}.$$

On the one hand, S can be viewed as the surface of revolution of the curve $\gamma : [0, 2\pi) \to \mathbb{R}^2$ given by $\gamma(t) = (2 + \cos(t), \sin(t))$. On the other hand, let

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \neq 0\}$$

and $F(x, y, z) = \left(\sqrt{x^2 + y^2} - 2\right)^2 - z^2$. Then $S = F^{-1}(1)$ with

$$\nabla F(x, y, z) = \left(\frac{2x}{\sqrt{x^2 + y^2}}\left(\sqrt{x^2 + y^2} - 2\right), \frac{2y}{\sqrt{x^2 + y^2}}\left(\sqrt{x^2 + y^2} - 2\right), 2z\right).$$

So that if $\nabla F = 0$ then $z = 0$ and either $\sqrt{x^2 + y^2} = 2$ or $x = y = 0$. Therefore, $\nabla F \neq 0$ on S and thus S is a regular surface by Proposition 3.2.6.

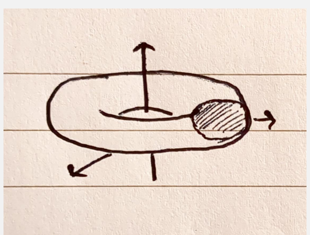
3. *Let $\Omega = \mathbb{R}^3$, and let $F : \mathbb{R}^3 \to \mathbb{R}$ be given by*

$$F(x, y, z) = yz.$$

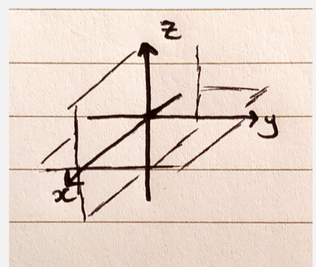
Then

$$\nabla F(x, y, z) = (0, z, y),$$

and so Proposition 3.2.6 does not apply to $F^{-1}(0) = \{y = 0\} \cup \{z = 0\}$.



(a) The torus.



(b) Statement 3 of Example 3.2.8 is not a surface as it is self-intersecting.

Figure 3.2.3: Surfaces as level sets.

Proposition 3.2.9. *Let $S \subseteq \mathbb{R}^3$ be a regular surface. Then for $p \in S$ there is a neighbourhood $V \subseteq S$ of p such that there is a smooth function f so that one of the following holds.*

1. *$V = \{z = f(x, y)\}$.*
2. *$V = \{y = f(x, z)\}$.*
3. *$V = \{x = f(y, z)\}$.*

Proof. Let $\phi : U \to S$ be a chart at p, where $U \subseteq \mathbb{R}^2$ is open. In particular, let

$$\phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

and $q = \phi^{-1}(p) \in U$. Then,

$$\mathrm{d}\phi_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

is injective and thus has rank two. Hence, a 2×2 minor of $\mathrm{d}\phi_q$ is invertible. Without loss of generality suppose

that
$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$
is invertible. Let $g : U \to \mathbb{R}^2$ be given by
$$g(u, v) = (x(u, v), y(u, v)).$$
By construction we can apply Theorem 3.2.5 to g to q to deduce there exists an open neighbourhood $V \subseteq U$ of q such that $g : V \to g(V)$ is a smooth diffeomorphism. Then $W = \phi(V)$ is an open neighbourhood of $p = \phi(q)$ in S which is the graph of $f : g(V) \to \mathbb{R}$ given by
$$(x, y) \xrightarrow{g^{-1}} (u, v) \mapsto z(u, v).$$
In particular, $f = z \circ g^{-1}$, which is smooth by the chain rule. $\square$

Remark 3.2.10. *From Proposition 3.2.6 we have that regular level sets are regular surfaces. With Proposition 3.2.9 we have that a regular surface is locally a regular level set. Indeed, for $p \in S$ let $V \subseteq S$ be the open neighbourhood of p given by Proposition 3.2.9. Assume without loss of generality that $V = \{z = f(x, y)\}$. Let $G : V \to \mathbb{R}$ be given by $G(x, y, z) = f(x, y) - z$. Then G is smooth as f is smooth, and $G^{-1}(0) = V$. Moreover,*
$$\nabla G(x, y, z) = (f_x(x, y), f_y(x, y), 1) \neq 0,$$
meaning V is a regular level set.

Example 3.2.11. *Consider the set*
$$S = \left\{(x, y, z) \in \mathbb{R}^3 : z = \sqrt{|y|}\right\}.$$
If S were a regular surface then locally it is a regular level set. In particular, around the origin, S is the graph of a smooth function.

- *Suppose S is the graph of a function in x, y, that is locally around the origin we have $(x, y, f(x, y))$. By the uniqueness of such a representation, it follows that $f(x, y) = \sqrt{|y|}$, however, f is not smooth.*
- *Suppose S is the graph of a function in x, z, that is locally around the origin we have $(x, f(x, z), z)$. However, in any neighbourhood of the origin the points $\left(x, z^2, z\right)$ and $\left(x, -z^2, z\right)$ lie on S. Therefore, $f(x, z)$ is not a well-defined function. Similarly, one shows that locally around the origin the set S is not a graph of a function in y, z.*

From the above observations, it follows that S is not a regular surface.

3.3 Tangent Vectors and Tangent Planes

Let $S \subseteq \mathbb{R}^3$ be a regular surface. For $p \in S$ there exists a linear space that best approximates the surface S at p.

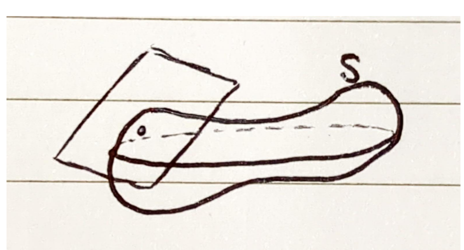


Figure 3.3.1: The best approximating plane to a surface at a point.

We can use these approximating surfaces to find the derivatives of functions of the form $f : S_1 \to S_2$ where S_1 and S_2 are regular surfaces.

Definition 3.3.1. *Let $S \subseteq \mathbb{R}^3$ be a regular surface and let $p \in S$.*

1. *A vector $v \in \mathbb{R}^3$ is a tangent vector to S at p if there exists a smooth map $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.*
2. *The tangent plane of S at p is*

$$T_pS = \{\gamma'(0) : \gamma(-\epsilon, \epsilon) \to S, \text{ smooth with } \gamma(0) = p\}.$$

That is, T_pS is the set of all tangent vectors to S at p.

Remark 3.3.2.

1. *Note that γ is not required to be a regular curve. This is so that the zero vector can be a tangent vector to the surface. Indeed, the zero vector is always in the tangent plane of a surface at a point as one can just take the curve $\gamma \equiv p$. The affine plane $T_pS + p$ is taken to be the plane that best approximates S at p.*
2. *Note statement 2 of Definition 3.3.1 is independent of the chart used to describe S, which makes it a difficult characterisation to compute the tangent plane in practice.*

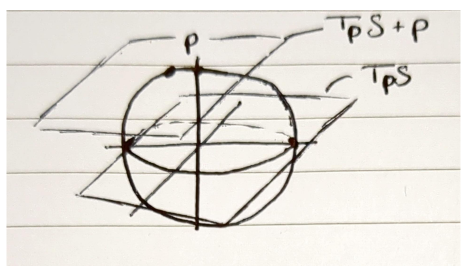


Figure 3.3.2: Using $T_pS + p$ to approximate a surface S at a point p.

Theorem 3.3.3. *Let $S \subseteq \mathbb{R}^3$ be a regular surface. Let $p \in S$ and $\phi : U \to S$ be a chart with $\phi(q) = p$. Then*

$$\mathrm{d}\phi_q\left(\mathbb{R}^2\right) = \operatorname{span}\left(\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right) = T_pS.$$

In particular, T_pS is a two-dimensional vector space.

Proof. Note that $\mathrm{d}\phi_q\left(\mathbb{R}^2\right)$ is spanned by the columns of

$$\mathrm{d}\phi_q = \begin{pmatrix} | & | \\ \frac{\partial \phi}{\partial u}(q) & \frac{\partial \phi}{\partial v}(q) \\ | & | \end{pmatrix}.$$

For any $v = a\frac{\partial \phi}{\partial u}(q) + b\frac{\partial \phi}{\partial v}(q) \in \mathrm{d}\phi_q\left(\mathbb{R}^2\right)$, let $q = (u_0, v_0)$ and consider $\alpha : (-\epsilon, \epsilon) \to S$ given by

$$\alpha(t) = \phi(u_0 + at, v_0 + bt).$$

Then α is smooth with $\alpha(0) = p$. Moreover,

$$\begin{aligned}
\alpha'(0) &= \frac{\partial \phi}{\partial u}(u_0, v_0)\frac{\partial u}{\partial t}(0) + \frac{\partial \phi}{\partial v}(u_0, v_0)\frac{\partial v}{\partial t}(0) \\
&= \frac{\partial \phi}{\partial u}(q)a + \frac{\partial \phi}{\partial v}(q)b \\
&= v.
\end{aligned}$$

Therefore, $v \in T_pS$ and thus $\mathrm{d}\phi_q\left(\mathbb{R}^2\right) \subseteq T_pS$. On the other hand, by Proposition 3.2.9, we can assume without loss of generality that S is given by $z = f(x, y)$ near p. Therefore, $\phi : U \to S$ is of the form

$$\phi(u, v) = (x(u, v), y(u, v), f(x(u, v), y(u, v))).$$

Let $q = \phi^{-1}(p) = (u_0, v_0)$. Moreover, consider $F : U \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ given by

$$F(u, v, t) = (x(u, v), y(u, v), f(x(u, v), y(u, v)) + t)\,.$$

Then $F(u_0, v_0, 0) = p$ and

$$\mathrm{d}F_{(u_0,v_0,0)} = \begin{pmatrix} x_u & x_v & 0 \\ y_u & y_v & 0 \\ f_x x_u + f_y y_u & f_x x_v + f_y y_v & 1 \end{pmatrix}.$$

Since $\mathrm{d}\phi_q$ is invertible, the matrix

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

is invertible, from the proof of Proposition 3.2.9. Thus, $\mathrm{d}F_{(u_0,v_0,0)}$ is invertible and so Theorem 3.2.5 can be applied to F at $(u_0, v_0, 0)$ to obtain $G = F^{-1}$ a smooth homeomorphism near p. In particular, there are smooth functions $u(x, y)$ and $v(x, y)$ such that

$$G(x, y, f(x, y)) = (u(x, y), v(x, y), 0).$$

Assume that $\alpha'(0) \in T_pS$ for $\alpha : (-\epsilon, \epsilon) \to S$ a smooth map with $\alpha(0) = p$. Let $\alpha(t) = (x(t), y(t), v(x(t), y(t)))$ then

$$\alpha(t) = \phi(u(x(t), y(t)), v(x(t), y(t))).$$

So by the chain rule it follows that

$$\alpha'(0) = \frac{\partial \phi}{\partial u}(q)\frac{\partial u}{\partial t}(0) + \frac{\partial \phi}{\partial v}(q)\frac{\partial v}{\partial t}(0).$$

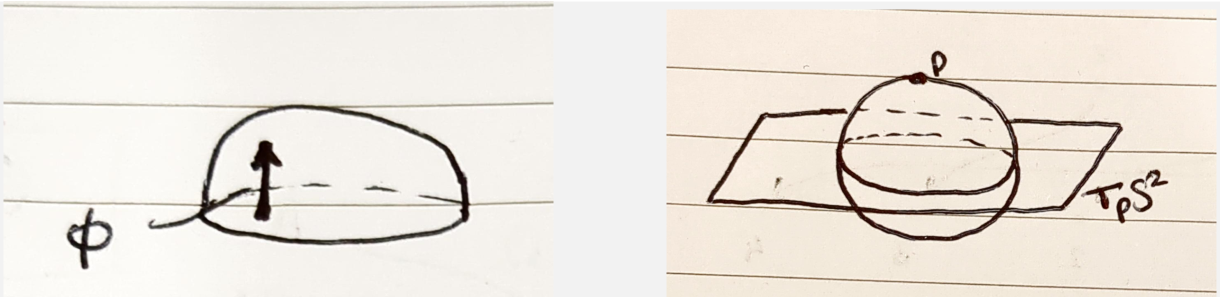
Thus $\alpha'(0) \in \mathrm{d}\phi_q\left(\mathbb{R}^2\right)$ and so $T_pS \subseteq \mathrm{d}\phi_q\left(\mathbb{R}^2\right)$. In conclusion, $\mathrm{d}\phi_q\left(\mathbb{R}^2\right) = T_pS$. □

Example 3.3.4. *Consider* $\mathrm{S}^2 = \left\{x^2 + y^2 + z^2 = 1\right\} \subseteq \mathbb{R}^3$ *and* $p = (0, 0, 1) \in \mathrm{S}^2$. *Let* $\phi : \left\{u^2 + v^2 \leq 1\right\} \to \mathrm{S}^2$ *be given by*

$$\phi(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right)$$

such that $\phi(0, 0) = (0, 0, 1)$. *Then*

$$\begin{aligned}
T_p\mathrm{S}^2 &= \operatorname{span}\left(\frac{\partial \phi}{\partial u}(0, 0), \frac{\partial \phi}{\partial v}(0, 0)\right) \\
&= \operatorname{span}\left(\left(1, 0, \frac{-u}{\sqrt{1 - u^2 - v^2}}\right)\Big|_{u=v=0}, \left(0, 1, \frac{-v}{\sqrt{1 - v^2 - u^2}}\right)\Big|_{u=v=0}\right) \\
&= \operatorname{span}((1, 0, 0), (0, 1, 0)) \\
&= \{z = 0\}.
\end{aligned}$$



(a) The chart ϕ used to calculate the tangent plane.

(b) The tangent plane for S at p.

Figure 3.3.3

Proposition 3.3.5. *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be smooth, with $\nabla F \neq 0$ on $S = F^{-1}(0)$. Then for any $p \in S$ we have*

$$T_pS = \left\{v \in \mathbb{R}^3 : \langle \nabla F(p), v\rangle = 0\right\} = (\nabla F(p))^{\perp}.$$

Proof. Let $\gamma : (-\epsilon, \epsilon) \to S$ be a smooth map with $\gamma(0) = p$. Then $F(\gamma(t)) = 0$ for all $t \in (-\epsilon, \epsilon)$, which means that $\frac{\mathrm{d}F(\gamma(t))}{\mathrm{d}t} = 0$. Hence,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(F(\gamma(t))\right)\big|_{t=0} \\ &= \langle \nabla F(\gamma(0)), \gamma'(0)\rangle \\ &= \langle \nabla F(p), \gamma'(0)\rangle . \end{aligned}$$

Hence, $T_pS \subseteq (\nabla F(p))^{\perp}$. Therefore, as T_pS and $(\nabla F(p))^{\perp}$ are two-dimensional vector spaces, it follows that $T_pS = (\nabla F(p))^{\perp}$. ☐

Remark 3.3.6. *The set $(\nabla F(p))^{\perp}$ is referred to as the orthogonal complement.*

Example 3.3.7. *Let*

$$S = \left\{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\right\} = F^{-1}(0)$$

where $F(x, y, z) = x^2 + y^2 - z$. Let $p = (1, 3, 10)$. As

$$\nabla F(x, y, z) = (2x, 2y, -1)$$

we have

$$\nabla F(p) = (2, 6, -1),$$

so that by Proposition 3.3.5 it follows that

$$T_pS = (\nabla F(p))^{\perp} = \left\{(x, y, z) \in \mathbb{R}^3 : 2x + 6y - z = 0\right\}.$$

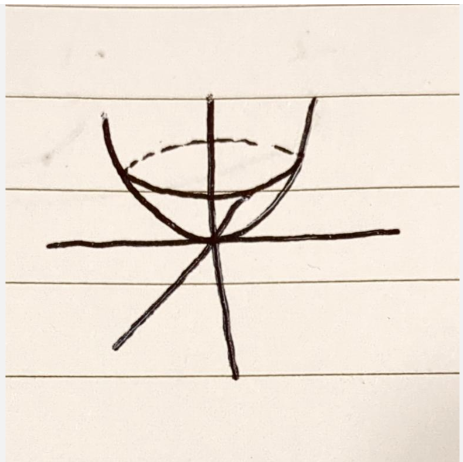


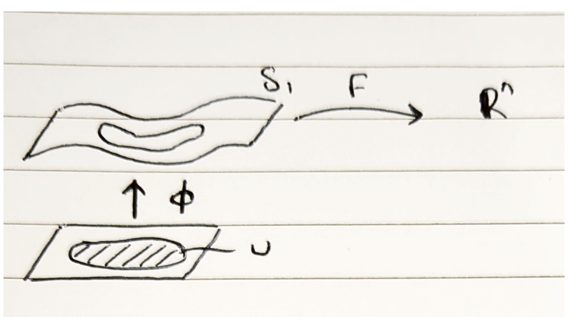
Figure 3.3.4: Paraboloid

3.4 Smooth Maps on Surfaces

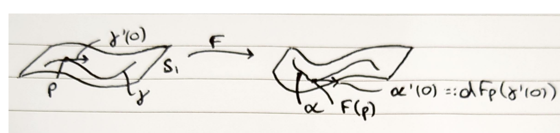
Definition 3.4.1. *Let $S_1, S_2 \subseteq \mathbb{R}^3$ be regular surfaces.*

1. *A map $F : S_1 \to \mathbb{R}^n$ is smooth if $F \circ \phi : U \to \mathbb{R}^n$ is smooth for every chart $\phi : U \to S_1$ for S_1.*
2. *A map $F : S_1 \to S_2$ is smooth if it is smooth when considered as a map into $\mathbb{R}^3$.*
3. *The differential of $F : S_1 \to S_2$ at $p \in S_1$ is the map $\mathrm{d}F_p : T_pS_1 \to T_{F(p)}S_2$ given as follows. For $\gamma'(0) \in T_pS_1$, where $\gamma : (-\epsilon, \epsilon) \to S_1$ is a smooth map with $\gamma(0) = p$ note that $\alpha(t) := F(\gamma(t)) : (-\epsilon, \epsilon) \to S_2$ is a smooth map with $\alpha(0) = F(\gamma(0)) = F(p)$. Let*

$$\mathrm{d}F_p\left(\gamma'(0)\right) := \alpha'(0) = \frac{\mathrm{d}F(\gamma(t))}{\mathrm{d}t}\Big|_{t=0}.$$



(a) The composition $F \circ \phi$ is a map from an open subset of $\mathbb{R}^2$ into $\mathbb{R}^n$ for which we have a notion of smoothness.



(b) The differential map considers the tangent of curves.

Figure 3.4.1

Exercise 3.4.2. *Let $S \subseteq \mathbb{R}^3$ be a regular surface. For $p \in S$, if $F \circ \phi : U \to \mathbb{R}^n$ is smooth for a chart $\phi : U \to S$ of p, show that $F : S \to \mathbb{R}^n$ is smooth as per statement 1 of Definition 3.4.1.*

Proposition 3.4.3. *Statement 3 of Definition 3.4.1 is independent of the choice of γ. That is, if $\gamma_1, \gamma_2 : (-\epsilon, \epsilon) \to S_1$ are such that $\gamma_1(0) = \gamma_2(0) = p$ and $\gamma_1'(0) = \gamma_2'(0)$ then*

$$\mathrm{d}F_p\left(\gamma_1'(0)\right) = \mathrm{d}F_p\left(\gamma_2'(0)\right).$$

Proof. Given the chart $\phi : U \to S_1$ for S_1 at p consider the smooth maps $t \mapsto (u_1(t), v_1(t))$ and $t \mapsto (u_2(t), v_2(t))$ as constructed in the proof of Theorem 3.3.3 such that

$$\gamma_1(t) = \phi(u_1(t), v_1(t))$$

and

$$\gamma_2(t) = \phi(u_2(t), v_2(t)).$$

Let $\phi^{-1}(p) = q$, then

$$\gamma_1'(0) = \frac{\partial \phi}{\partial u}(q)\frac{\partial u_1}{\partial t}(0) + \frac{\partial \phi}{\partial v}(q)\frac{\partial v_1}{\partial t}(0)$$

and

$$\gamma_2'(0) = \frac{\partial \phi}{\partial u}(q)\frac{\partial u_2}{\partial t}(0) + \frac{\partial \phi}{\partial v}(q)\frac{\partial v_2}{\partial t}(0).$$

Since $\gamma_1'(0) = \gamma_2'(0)$ and $\left\{\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right\}$ are linearly independent, it follows that $u_1'(0) = u_2'(0)$ and $v_1'(0) = v_2'(0)$. That is, (u_1, v_1) and (u_2, v_2) have the same tangent vector at $t = 0$. On the other hand, as $F \circ \phi : \mathbb{R}^2 \to \mathbb{R}^3$ is smooth the chain rule can be applied to obtain

$$\begin{aligned}
\mathrm{d}F_p\left(\gamma_i'(0)\right) &= (F \circ \gamma_i)'(0) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}((F \circ \phi)(u_i(t), v_i(t)))|_{t=0} \\
&= \frac{\partial (F \circ \phi)}{\partial u}(q)\frac{\partial u_i}{\partial t}(0) + \frac{\partial (F \circ \phi)}{\partial v}(q)\frac{\partial v_i}{\partial t}(0).
\end{aligned}$$

Thus, $\mathrm{d}F_p\left(\gamma_1'(0)\right) = \mathrm{d}F_p\left(\gamma_2'(0)\right)$ meaning $\mathrm{d}F_p$ is independent of the choice of γ. □

Proposition 3.4.4. *Let S_1, S_2 be regular surfaces. For $F : S_1 \to S_2$ and $p \in S_1$ the differential $\mathrm{d}F_p : T_pS_1 \to T_{F(p)}S_2$ is a linear map.*

Proof. Let $v = \gamma_1'(0)$ and $w = \gamma_2'(0)$ be vectors in T_pS_1 with $\gamma_1, \gamma_2 : (-\epsilon, \epsilon) \to S_1$. Let $\phi : U \to S_1$ be a chart for S_1 at p. Then, as in the proof of Proposition 3.4.3, there are smooth maps $t \mapsto (u_1(t), v_1(t))$ and $t \mapsto (u_2(t), v_2(t))$ such that

$$\gamma_1(t) = \phi(u_1(t), v_1(t))$$

and

$$\gamma_2(t) = \phi(u_2(t), v_2(t)).$$

Without loss of generality let $\phi(0,0) = p$. Let $\lambda \in \mathbb{R}$ and consider $\gamma_3 : (-\epsilon, \epsilon) \to S_1$ given by

$$\gamma_3(t) = \phi(u_1(t) + \lambda u_2(t), v_1(t) + \lambda v_2(t)).$$

Then $\gamma_3(0) = p$ and

$$\begin{aligned}
\gamma_3'(0) &= \frac{\partial \phi}{\partial u}(0,0)\left(u_1'(0) + \lambda u_2'(0)\right) + \frac{\partial \phi}{\partial v}(0,0)\left(v_1'(0) + \lambda v_2'(0)\right) \\
&= \gamma_1'(0) + \lambda \gamma_2'(0) \\
&= v + \lambda w.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{d}F_p(v+\lambda w) &= \mathrm{d}F_p\left(\gamma_3'(0)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left((F\circ\phi)(u_1(t)+\lambda u_2(t), v_1(t)+\lambda v_2(t))\right)|_{t=0} \\
&= \frac{\partial(F\circ\phi)}{\partial u}(0,0)\left(u_1'(0)+\lambda u_2'(0)\right)+\frac{\partial(F\circ\phi)}{\partial v}(0,0)\left(v_1'(0)+\lambda v_2'(0)\right) \\
&= \left(\frac{\partial(F\circ\phi)}{\partial u}(0,0)u_1'(0)+\frac{\partial(F\circ\phi)}{\partial v}(0,0)v_1'(0)\right) \\
&\quad+\lambda\left(\frac{\partial(F\circ\phi)}{\partial u}(0,0)u_2'(0)+\frac{\partial(F\circ\phi)}{\partial v}(0,0)v_2'(0)\right). \qquad (3.4.1)
\end{aligned}$$

Using the chain rule observe that

$$\begin{aligned}
\frac{\partial(F\circ\phi)}{\partial u}(0,0)u_1'(0)+\frac{\partial(F\circ\phi)}{\partial v}(0,0)v_1'(0) &= \frac{\mathrm{d}}{\mathrm{d}t}((F\circ\phi)(u_1(t),v_1(t)))|_{t=0} \\
&= \mathrm{d}F_p\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi(u_1(t),v_1(t))|_{t=0}\right) \\
&= \mathrm{d}F_p\left(\gamma_1'(0)\right) \\
&= \mathrm{d}F_p(v).
\end{aligned}$$

Similarly,

$$\frac{\partial(F\circ\phi)}{\partial u}(0,0)u_2'(0)+\frac{\partial(F\circ\phi)}{\partial v}(0,0)v_2'(0)=\mathrm{d}F_p(w).$$

Therefore, (3.4.1) becomes

$$\mathrm{d}F_p(v+\lambda w)=\mathrm{d}F_p(v)+\lambda\mathrm{d}F_p(w),$$

which shows that $\mathrm{d}F_p$ is linear. ☐

If $S\subseteq\mathbb{R}^3$ is a regular surface, then the differential of $f:S\to\mathbb{R}$ at $p\in S$ is $\mathrm{d}f_p:T_pS\to\mathbb{R}$ given by

$$\mathrm{d}f_p\left(\gamma'(0)\right)=\frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}t}\Big|_{t=0}.$$

All results for the differential of smooth maps between surfaces hold for $\mathrm{d}f_p$, namely $\mathrm{d}f_p$ is well-defined and linear.

Example 3.4.5. *Recall that for* $\mathrm{S}^2=\left\{x^2+y^2+z^2=1\right\}\subseteq\mathbb{R}^3$ *and* $p=(0,0,1)$*, we have* $T_p\mathrm{S}^2=\{z=0\}$*. The chart* $\phi:\left\{u^2+v^2\le 1\right\}\to\mathrm{S}^2$ *given by*

$$\phi(u,v)=\left(u,v,\sqrt{1-u^2-v^2}\right)$$

is such that $p=\phi(0,0)$*. Let* $f:\mathrm{S}^2\to\mathbb{R}$ *be given by* $f(x,y,z)=y$*. Then* $f\circ\phi:\left\{u^2+v^2\le 1\right\}\to\mathbb{R}$ *is given by*

$$(f\circ\phi)(u,v)=v,$$

which means that f *is smooth. The vectors* $\left\{\frac{\partial\phi}{\partial u}(0,0),\frac{\partial\phi}{\partial v}(0,0)\right\}$ *form a basis for* $T_p\mathrm{S}^2$*. They are the tangent vectors to the curves* $t\mapsto\phi(t,0)$ *and* $t\mapsto\phi(0,t)$ *respectively. Thus*

$$\mathrm{d}f_p\left(\frac{\partial\phi}{\partial u}(0,0)\right)=\frac{\mathrm{d}(f\circ\phi)(t,0)}{\mathrm{d}t}\Big|_{t=0}=0$$

and

$$\mathrm{d}f_p\left(\frac{\partial\phi}{\partial v}(0,0)\right)=\frac{\mathrm{d}(f\circ\phi)(0,t)}{\mathrm{d}t}\Big|_{t=0}=1.$$

In particular, if $w \in T_p\mathrm{S}^2$ we can write

$$w = w_1 \frac{\partial \phi}{\partial u}(0,0) + w_2 \frac{\partial \phi}{\partial v}(0,0) = (w_1, w_2, 0)$$

to deduce that $\mathrm{d}f_p(w) = w_2$.

3.5 Normal Vectors and the Gauss Map

Recall the importance of the normal vector in defining the curvature of a curve. Thus, to start discussing the curvature of a surface, it will be useful to investigate the normal vectors to surfaces.

Definition 3.5.1. *Let $S \subseteq \mathbb{R}^3$ be a regular surface. Given $p \in S$, a normal vector $N \in \mathbb{R}^3$ to S at p is such that*

$$\langle N, v \rangle = 0$$

for all $v \in T_pS$. In particular, N is a unit normal vector if $\langle N, N \rangle = 1$.

Definition 3.5.2. *Let $S \subseteq \mathbb{R}^3$ be a regular surface. Then a smooth map $N : S \to \mathrm{S}^2 \subseteq \mathbb{R}^3$ is a unit normal field if $N(p)$ is a unit normal to S at p for every $p \in S$.*

Locally, unit normal fields always exist. If $\phi : U \to S$ is a chart, with $\phi(q) = p$, then $\left\{\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right\}$ spans T_pS. Thus,

$$N(p) = \frac{\frac{\partial \phi}{\partial u}(q) \times \frac{\partial \phi}{\partial v}(q)}{\left|\frac{\partial \phi}{\partial u}(q) \times \frac{\partial \phi}{\partial v}(q)\right|}$$

is a unit normal field. Similarly, if $S = F^{-1}(c)$ for some smooth function $F : \mathbb{R}^3 \to \mathbb{R}$ with $\nabla F(p) \neq 0$ and $c \in \mathbb{R}$, then $T_pS = (\nabla F(p))^{\perp}$ and so

$$N(p) = \frac{\nabla F(p)}{|\nabla F(p)|}$$

is a unit normal field. In this case, the unit normal field exists globally, however, this is not always the case.

Definition 3.5.3. *A regular surface $S \subseteq \mathbb{R}^3$ is orientable if there exists a unit normal field $N : S \to \mathrm{S}^2$.*

Remark 3.5.4. *The tangent plane at a point of a regular surface $S \subseteq \mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$. Consequently, there exist two unit normal vectors to the tangent plane. For regular level sets the canonical choice is $\nabla F(p)$. In other cases, we refer to the choice as an orientation of the surface. In particular, an oriented surface is an orientable surface for which the choice of normal has been made.*

Definition 3.5.5. *For an oriented surface S, the map $N : S \to \mathrm{S}^2$ is referred to as the Gauss map of S.*

Example 3.5.6.

1. *The Möbius strip is a non-orientable surface.*

2. *Recall that $\mathrm{S}^2 = F^{-1}(1)$ where $F : \mathbb{R}^3 \to \mathbb{R}$ is given by $F(x,y,z) = x^2 + y^2 + z^2$. Therefore, S^2 is orientable. In particular,*

$$N(x,y,z) = \frac{\nabla F(x,y,z)}{|\nabla F(x,y,z)|} = (x,y,z).$$

Hence, the Gauss map for S^2 is the identity map.

3. *Consider $S = \{ax+by+cz=d\} \subseteq \mathbb{R}^3$ for some $a,b,c,d \in \mathbb{R}$. Then $S = F^{-1}(d)$ for $F : \mathbb{R}^3 \to \mathbb{R}$ given by $F(x,y,z) = ax+by+cz$. Thus,*

$$N(x,y,z) = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}},$$

which is constant.

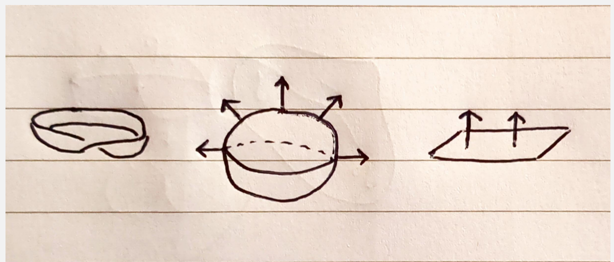


Figure 3.5.1: The Möbius strip is not orientable as a global normal unit field will not be continuous, it will flip along a boundary. The sphere and the plane are orientable surfaces.

Exercise 3.5.7. *Show that the Gauss map for S^2 obtained in statement 2 of Example 3.5.6 can similarly be obtained through the chart $\phi : \left\{u^2+v^2 \leq 1\right\} \to \mathrm{S}^2$ given by*

$$\phi(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right).$$

Proposition 3.5.8. *Let $S \subseteq \mathbb{R}^3$ be an orientable surface with Gauss map $N : S \to \mathrm{S}^2$. Then for each $p \in S$ we have $T_pS = T_{N(p)}\mathrm{S}^2$.*

Proof. Recall that $q \in \mathrm{S}^2$ is normal to $T_q\mathrm{S}^2$. That is,

$$T_{N(p)}\mathrm{S}^2 = \left\{v \in \mathbb{R}^3 : \langle N(p), v\rangle = 0\right\} = T_pS.$$

$\square$

The differential of N at $p \in S$, namely $\mathrm{d}N_p : T_pS \to T_{N(p)}\mathrm{S}^2$, is a linear map. In particular, from Proposition 3.5.8 we see that $\mathrm{d}N_p : T_pS \to T_pS$.

Example 3.5.9.

1. *Let*

$$\mathrm{S}^2_r := \left\{x^2+y^2+z^2 = r^2\right\} = F^{-1}\left(r^2\right),$$

where $F(x,y,z) = x^2+y^2+z^2$. Then,

$$N(x,y,z) = \frac{1}{r}(x,y,z).$$

Let $\gamma : (-\epsilon, \epsilon) \to \mathrm{S}^2_r$ be a regular curve with $\gamma(0) = p$. Then

$$\begin{aligned}
\mathrm{d}N_p\left(\gamma'(0)\right) &= \frac{\mathrm{d}N(\gamma(t))}{\mathrm{d}t}\Big|_{t=0} \\
&= \frac{1}{r}\frac{\mathrm{d}(\gamma(t))}{\mathrm{d}t}\Big|_{t=0} \\
&= \frac{1}{r}\gamma'(0).
\end{aligned}$$

Thus, $\mathrm{d}N_p : T_p\mathrm{S}^2_r \to T_p\mathrm{S}^2_r$ is $\mathrm{d}N_p = \frac{1}{r}\mathrm{id}$.

2. *For the plane $S = \{ax + by + cz = d\} \subseteq \mathbb{R}^3$ we have $N = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$ and so $\mathrm{d}N_p = 0$.*

These suggest a connection between $\mathrm{d}N_p$ and the curvature of a surface, as the map $\mathrm{d}N_p$ conforms with our intuition of the curvature of a surface.

The derivative of the Gauss map, $\mathrm{d}N_p : T_pS \to T_{N(p)}\mathrm{S}^2$, can be viewed from different perspectives.

1. As Proposition 3.5.8 notes, the tangent plane $T_{N(p)}\mathrm{S}^2$ consists of the vectors orthogonal to $N(p)$, thus it is the same as T_pS. Therefore, we can view the derivative as a map $\mathrm{d}N_p : T_pS \to T_pS$.
2. For $\gamma'(0) \in T_pS$, since N is the unit normal we must have $\langle N(\gamma(t)), N(\gamma(t))\rangle = 1$. Consequently,

$$\begin{aligned}
\langle \mathrm{d}N_p(\gamma'(0)), N_p(\gamma(0))\rangle &= \left\langle \frac{\mathrm{d}}{\mathrm{d}t}N_p(\gamma(t))\big|_{t=0}, N_p(\gamma(0))\right\rangle \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle N_p(\gamma(t)), N_p(\gamma(t))\rangle\big|_{t=0} \\
&= 0.
\end{aligned}$$

Thus, $\mathrm{d}N_p(\gamma'(0))$ is orthogonal to $N(\gamma(0))$, meaning $\mathrm{d}N_p(\gamma'(0)) \in T_pS$.

Proposition 3.5.10. *Let $S \subseteq \mathbb{R}^3$ be a non-empty, compact, connected and oriented regular surface. Then the Gauss map $N : S \to \mathrm{S}^2$ is surjective.*

Proof. Step 1: Characterise the normal vectors of S.
For $v \in \mathrm{S}^2$, let $f_v(x) = \langle x, v\rangle$. Suppose that f_v has a maximum at $p \in S$. For $w \in T_pS$ there exists a smooth map $\gamma : (-\epsilon, \epsilon) \to S$ with $\gamma(0) = p$ and $\gamma'(0) = w$. Let $f := f_v \circ \gamma$. Then f has a maximum at $t = 0$ as $\gamma(0) = p$. It follows that

$$0 = \frac{\mathrm{d}f}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma(t), v\rangle\Big|_{t=0} = \langle \gamma'(0), v\rangle = \langle w, v\rangle,$$

which implies that $v \in (T_pS)^\perp$, and so v is normal to S at p.
Step 2: Show that v as characterised in step 1 is an outward normal vector to S.
For $v \in \mathrm{S}^2$ suppose that f_v is maximized at $p \in S$, that is

$$\langle x, v\rangle \leq \langle p, v\rangle =: c$$

for all $x \in S$. Note that $\left\{x \in \mathbb{R}^3 : \langle x, v\rangle = c\right\}$ is a plane containing p that divides $\mathbb{R}^3$. In particular,

$$S \subseteq \left\{x \in \mathbb{R}^3 : \langle x, v\rangle \leq c\right\},$$

meaning v is an outward normal to S.
Step 3: Deduce that N is surjective.
Observe that for any $v \in \mathrm{S}^2$ we have

$$\begin{aligned}
|f_v(s_1) - f_v(s_2)| &= |\langle s_1, v\rangle - \langle s_2, v\rangle| \\
&= |\langle s_1 - s_2, v\rangle| \\
&\overset{(1)}{\leq} |s_1 - s_2||v| \\
&= |s_1 - s_2|,
\end{aligned}$$

where (1) is an application of the Cauchy-Schwartz inequality. Hence, we see that f_v is continuous. In particular, as S is compact this means that the function $f_v : S \to \mathbb{R}$ achieves its maximum at some point $p \in S$. We know from steps 1 and 2 that v is an outward normal to S at p. Therefore, $N(p) = v$ which implies that N is surjective. □

3.6 Second Fundamental Form

Throughout, let S be an oriented surface with $p \in S$.

Definition 3.6.1. *The second fundamental form of S at p is the map $A_p : T_pS \times T_pS \to \mathbb{R}$ given by*

$$A_p(x, y) = -\langle x, \mathrm{d}N_p(y)\rangle,$$

where $N : S \to \mathrm{S}^2$ is the Gauss map of S.

Example 3.6.2.

1. *For S^2_r, the sphere of radius r, as $N = \frac{1}{r}\mathrm{id}$ we have that*

$$A_p(x, y) = -\frac{1}{r}\langle x, y\rangle$$

   for all $p \in \mathrm{S}^2_r$.

2. *For S a plane, as N is constant we have that $A_p(x, y) \equiv 0$ for all $p \in S$.*

Proposition 3.6.3. *The second fundamental form $A_p : T_pS \times T_pS \to \mathbb{R}$ is a symmetric bilinear map.*

Proof. Observe that

$$\begin{aligned} A_p(x, \lambda_1 y + \lambda_2 z) &= -\langle x, \mathrm{d}N_p(\lambda_1 y + \lambda_2 z)\rangle \\ &= -\langle x, \lambda_1 \mathrm{d}N_p(y) + \lambda_2 \mathrm{d}N_p(z)\rangle \\ &= -\lambda_1\langle x, \mathrm{d}N_p(y)\rangle - \lambda_2\langle x, \mathrm{d}N_p(z)\rangle \\ &= \lambda_1 A_p(x, y) + \lambda_2 A_p(x, z), \end{aligned}$$

and

$$\begin{aligned} A_p(\lambda_1 x + \lambda_2 y, z) &= -\langle \lambda_1 x + \lambda_2 y, \mathrm{d}N_p(z)\rangle \\ &= -\lambda_1\langle x, \mathrm{d}N_p(z)\rangle - \lambda_2\langle y, \mathrm{d}N_p(z)\rangle \\ &= \lambda_1 A_p(x, z) + \lambda_2 A_p(y, z), \end{aligned}$$

meaning A_p is bilinear. Let $\phi : U \to S$ be a chart for S at p, and let $q = \phi^{-1}(p)$. Then

$$\begin{aligned} A_p\left(\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right) &= -\left\langle \frac{\partial \phi}{\partial u}(q), \mathrm{d}N_p\left(\frac{\partial \phi}{\partial v}(q)\right)\right\rangle \\ &= -\left\langle \frac{\partial \phi}{\partial u}(q), \frac{\partial (N \circ \phi)}{\partial v}(q)\right\rangle. \end{aligned} \tag{3.6.1}$$

Since $N \circ \phi$ maps into vectors normal to S, and $\frac{\partial \phi}{\partial u}$ is in the tangent space of S we have

$$\left\langle \frac{\partial \phi}{\partial u}, N \circ \phi\right\rangle = 0.$$

Differentiating with respect to v it follows that

$$0 = \frac{\partial}{\partial v}\left\langle \frac{\partial \phi}{\partial u}, N \circ \phi \right\rangle = \left\langle \frac{\partial^2 \phi}{\partial v \partial u}, N \circ \phi \right\rangle + \left\langle \frac{\partial \phi}{\partial u}, \frac{\partial (N \circ \phi)}{\partial v} \right\rangle.$$

Similarly,

$$0 = \left\langle \frac{\partial^2 \phi}{\partial v \partial u}, N \circ \phi \right\rangle + \left\langle \frac{\partial \phi}{\partial v}, \frac{\partial (N \circ \phi)}{\partial u} \right\rangle,$$

where we have exchanged the order of differentiation in the first inner product as ϕ is smooth. Combined with (3.6.1) it follows that

$$\begin{aligned} A_p\left(\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right) &= -\left\langle \frac{\partial \phi}{\partial u}(q), \frac{\partial (N \circ \phi)}{\partial v}(q) \right\rangle \\ &= \left\langle \frac{\partial^2 \phi}{\partial v \partial u}(q), (N \circ \phi)(q) \right\rangle \\ &= -\left\langle \frac{\partial \phi}{\partial v}(q), \frac{\partial (N \circ \phi)}{\partial u}(q) \right\rangle \\ &= A_p\left(\frac{\partial \phi}{\partial v}(q), \frac{\partial \phi}{\partial u}(q)\right). \end{aligned}$$

Therefore, as $\left\{\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right\}$ forms a basis for $T_p(S)$ we deduce that A_p is symmetric. ☐

Remark 3.6.4. *If ϕ is a chart for S at p, then as $\left\{\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right\}$ is an orthogonal basis for T_pS we can recover $\mathrm{d}N_p(x)$ through*

$$\begin{aligned} \mathrm{d}N_p(x) &= \left\langle \frac{\partial \phi}{\partial u}(q), \mathrm{d}N_p(x) \right\rangle \frac{\partial \phi}{\partial u}(q) + \left\langle \frac{\partial \phi}{\partial v}(q), \mathrm{d}N_p(x) \right\rangle \frac{\partial \phi}{\partial v}(q) \\ &= -A_p\left(\frac{\partial \phi}{\partial u}(q), x\right) \frac{\partial \phi}{\partial u}(q) - A_p\left(\frac{\partial \phi}{\partial v}(q), x\right) \frac{\partial \phi}{\partial v}(q), \end{aligned}$$

for $x \in T_pS$.

Proposition 3.6.5. *The differential of the Gauss map $\mathrm{d}N_p : T_pS \to T_pS$ is diagonalisable with real eigenvalues. That is, there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ and an orthonormal basis $\{X_1, X_2\}$ for T_pS such that $\mathrm{d}N_p(X_1) = -\lambda_1 X_1$ and $\mathrm{d}N_p(X_2) = -\lambda_2 X_2$.*

Proof. Let $\{v, w\}$ be an orthonormal basis for T_pS with

$$M = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

being the associated matrix for $\mathrm{d}N_p$. That is,

$$\begin{cases} \mathrm{d}N_p(v) = a_1 v + a_2 w \\ \mathrm{d}N_p(w) = b_1 v + b_2 w. \end{cases}$$

Using Proposition 3.6.3 we have

$$\begin{aligned} a_2 &= \langle w, \mathrm{d}N_p(v) \rangle \\ &= -A_p(w, v) \\ &= -A_p(v, w) \\ &= \langle v, \mathrm{d}N_p(w) \rangle \\ &= b_1. \end{aligned}$$

Thus, M is a real symmetric matrix and thus diagonalisable with real eigenvalues. $\square$

Definition 3.6.6.

1. *The real numbers $\lambda_1 \leq \lambda_2$ from Proposition 3.6.5 are referred to as the principal curvatures of the surface S at p.*

2. *The orthonormal tangent vectors $\{X_1, X_2\}$ from Proposition 3.6.5 are referred to as the principal directions of S at p.*

Remark 3.6.7.

1. *One should understand λ_1 and λ_2 as functions of $p \in S$.*

2. *Note that*
$$A_p(X_1, X_1) = -\langle X_1, \mathrm{d}N_p(X_1)\rangle = \langle X_1, \lambda_1 X_1\rangle = \lambda_1 |X_1|^2 = \lambda_1(p).$$
Similarly, $A_p(X_2, X_2) = \lambda_2(p)$ and $A_p(X_1, X_2) = A_p(X_2, X_1) = 0$.

Proposition 3.6.8. *If $\lambda_1(p) \leq \lambda_2(p)$ are the principal curvatures at $p \in S$, then*
$$\lambda_1 = \min \{A_p(x, x) : x \in T_pS,\ |x| = 1\}$$
and
$$\lambda_2 = \max \{A_p(x, x) : x \in T_pS,\ |x| = 1\}.$$

Proof. For $x \in T_pS$ with $|x| = 1$, one can write $x = c_1 X_1 + c_2 X_2$ where $c_1^2 + c_1^2 = 1$. Then as A_p is bilinear it follows that
$$\begin{aligned} A_p(x, x) &= A_p(c_1 X_1 + c_2 X_2, c_1 X_1 + c_2 X_2) \\ &= c_1^2 A_p(X_1, X_1) + 2c_1 c_2 A_p(X_1, X_2) + c_2^2 A_p(X_2, X_2). \end{aligned}$$
Therefore,
$$A_p(x, x) = c_1^2 \lambda_1 + c_2^2 \lambda_2 \geq c_1^2 \lambda_1 + c_2^2 \lambda_1 = \lambda_1,$$
and
$$A_p(x, x) = c_1^2 \lambda_1 + c_2^2 \lambda_2 \leq c_1^2 \lambda_2 + c_2^2 \lambda_2 = \lambda_2.$$
With λ_1 and λ_2 being realised for $x = X_1$ and $x = X_2$ respectively. $\square$

Example 3.6.9.

1. *For S_r^2, the sphere of radius r, we have that $A_p(x, y) = -\frac{1}{r}\langle x, y\rangle$ for all $x, y \in T_pS$. Thus,*
$$\lambda_1(p) = \lambda_2(p) = -\frac{1}{r}$$
for all $p \in \mathrm{S}_r^2$. If one were to re-orientate the surface such that N points inward, the sign of the principal curvatures would switch.

2. *If S is a plane then $A_p(x, y) = 0$ for all $x, y \in T_pS$, and so*
$$\lambda_1(p) = \lambda_2(p) = 0$$
for all $p \in S$.

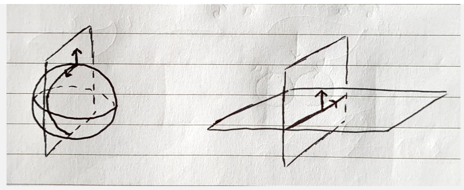


Figure 3.6.1: The intersecting planar curves for the sphere and the plane.

Exercise 3.6.10. *Suppose S is a connected surface with $\lambda_1(p) = \lambda_2(p) = 0$ for all $p \in S$. Show that S must lie in a plane.*

3.7 Normal Curvature

Let $\gamma : (-\epsilon, \epsilon) \to S$ be a regular curve parameterised by arc length in S with $\gamma(0) = p$. The curvature vector of γ at p is $\mathbf{k}(0) := \gamma''(0) \in \mathbb{R}^3$. In particular,

$$\mathbf{k}(0) = \langle \mathbf{k}(0), N(p) \rangle N(p) + \mathbf{k}_{\mathsf{tang}}(0),$$

where $\langle \mathbf{k}(0), N(p) \rangle N(p) \in (T_pS)^\perp$ and $\mathbf{k}_{\mathsf{tang}}(0) \in T_pS$. The curvature component perpendicular to S, namely $\langle \mathbf{k}(0), N(p) \rangle N(p)$, measures the curving of S at p in the direction of $\gamma'(0)$. We refer to $k_n(0) := \langle \mathbf{k}(0), N(p) \rangle$ as the normal curvature of S at p in the direction $\gamma'(0)$.

Theorem 3.7.1. *Let $S \subseteq \mathbb{R}^3$ be an orientable regular surface. Let $p \in S$ and $v \in T_pS$ with $|v| = 1$. Then, the normal curvature of S at p in the direction of v satisfies*

$$k_n(p) = A_p(v, v).$$

Proof. Let $\gamma : (-\epsilon, \epsilon) \to S$ be a regular curve parameterised by arc length, with $\gamma(0) = p$ and $\gamma'(0) = v$. Then for $t \in (-\epsilon, \epsilon)$ we have

$$\langle \gamma'(t), N(\gamma(t)) \rangle = 0.$$

Differentiating this with respect to t, and evaluating at $t = 0$ yields

$$\begin{aligned} 0 &= \langle \gamma''(0), N(\gamma(0)) \rangle + \left\langle \gamma'(0), \mathrm{d}N_{\gamma(0)} \left(\gamma'(0)\right) \right\rangle \\ &= \langle \mathbf{k}(0), N(p) \rangle + \langle v, \mathrm{d}N_p(v) \rangle \\ &= k_n(p) - A_p(v, v). \end{aligned}$$

□

Remark 3.7.2.

1. *A regular curve $\gamma : I \to S$ is an asymptotic line if the normal curvature at every point $\gamma(t)$ is zero. That is, $\langle \mathbf{k}(t), N(\gamma(t)) \rangle = 0$ for all $t \in I$.*
2. *Theorem 3.7.1 says that for any $x \in T_pS$ with $|x| = 1$, the planar curve*

$$S \cap \operatorname{span}\{x, N(p)\} \subseteq \operatorname{span}\{x, N(p)\}$$

has curvature $A_p(x, x)$ at p.

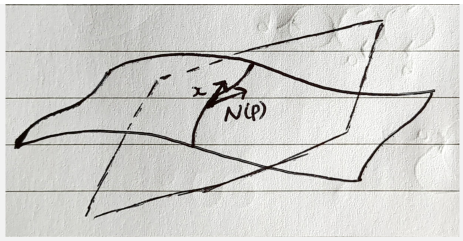


Figure 3.7.1

Thus, in conjunction with Proposition 3.6.8, it follows that the principal curvatures of an orientable surface are the minimum and maximum values of the normal curvature at p along all directions in T_pS respectively.

3.8 Gauss and Mean Curvature

Definition 3.8.1. *Let S be an oriented surface with Gauss map $N : S \to \mathrm{S}^2$.*

1. *The Gauss curvature of S is the map $K : S \to \mathbb{R}$ given by*

$$K(p) = \det(\mathrm{d}N_p) = \lambda_1(p)\lambda_2(p).$$

2. *The mean curvature of S is the map $H : S \to \mathbb{R}$ given by*

$$H(p) = -\frac{1}{2}\mathrm{tr}(\mathrm{d}N_p) = \frac{\lambda_1(p)+\lambda_2(p)}{2}.$$

Remark 3.8.2.

1. *The map K can be thought of as the geometric average of principal curvatures, whereas H is the arithmetic average.*
2. *The constructions of λ_1, λ_2, K and H are very much extrinsic. However, as an object K is an intrinsic object, which is made precise in Theorem 3.14.1.*

Example 3.8.3.

1. *The sphere of radius r, namely S^2_r, has $\lambda_1(p) = \lambda_2(p) = -\frac{1}{r}$ for all $p \in \mathrm{S}^2_r$. Thus,*

$$K(p) = \frac{1}{r^2}$$

and

$$H(p) = -\frac{1}{r}$$

for all $p \in \mathrm{S}^2_r$.

2. *The plane Π, has $\lambda_1(p) = \lambda_2(p) = 0$ for all $p \in \Pi$. So $K \equiv H \equiv 0$.*
3. *Consider the cylinder*

$$C_r := \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = r^2\right\} = F^{-1}\left(r^2\right)$$

where $F(x,y,z)=x^2+y^2$. Note that

$$\nabla F(x,y,z)=2(x,y,0),$$

and so $\nabla F(x,y,z)\neq 0$ on C_r. Hence, C_r is a regular level set. In particular,

$$N(x,y,z)=\frac{2(x,y,0)}{\sqrt{4\left(x^2+y^2\right)}}=\frac{1}{r}(x,y,0).$$

Given $v=(v_1,v_2,v_3)=\gamma'(0)\in T_{(x,y,z)}C_r$, where $\gamma:(-\epsilon,\epsilon)\to C_r$ is some smooth map, we have

$$\begin{aligned}\mathrm{d}N_{(x,y,z)}(v)&=\frac{\mathrm{d}(N(\gamma(t)))}{\mathrm{dt}}\Big|_{t=0}\\&=\frac{1}{r}\left(\gamma_1'(0),\gamma_2'(0),0\right)\\&=\frac{1}{r}(v_1,v_2,0),\end{aligned}$$

hence,

$$A_{(x,y,z,)}(v,w)=-\frac{1}{r}\left(v_1w_1+v_2w_2\right).$$

Observe that $X_1=(0,0,1)$ and $X_2=\frac{1}{r}(-y,x,0)$ are unit tangent vectors with

$$\mathrm{d}N_{(x,y,z)}(X_1)=0$$

and

$$\mathrm{d}N_{(x,y,z)}(X_2)=\frac{1}{r^2}(-y,x,0)=\frac{1}{r}X_2.$$

Therefore, X_1 and X_2 are the principal directions and the corresponding principal curvatures are $\lambda_1(p)=0$ and $\lambda_2(p)=\frac{1}{r}$. In particular, $K(p)=0$ and $H(p)=\frac{1}{2r}$ for all $p\in C_r$.

These surfaces are examples of constant mean curvature surfaces. Moreover, the plane is an example of a minimal surface as $H\equiv 0$.

4. *Consider the surface*

$$S=\left\{x^2-y^2=z\right\}=F^{-1}(0)$$

where $F(x,y,z)=x^2-y^2-z$. Observe that

$$\nabla F(x,y,z)=(2x,-2y,-1)\neq 0$$

on S. Thus,

$$N(x,y,z)=\frac{\nabla F}{|\nabla F|}=\frac{(2x,-2y,-1)}{\sqrt{4\left(x^2+y^2\right)+1}}.$$

In particular,

$$N(0)=(0,0,-1).$$

For $(v_1,v_2,v_3)\in T_0S$ let $\gamma:(-\epsilon,\epsilon)\to S$ be a smooth map such that $\gamma(0)=0$ and $\gamma'(0)=(v_1,v_2,v_3)$. Then

$$(N\circ\gamma)(t)=\frac{1}{\sqrt{4\left(\gamma_1(t)^2+\gamma_2(t)^2\right)+1}}\left(2\gamma_1(t),-2\gamma_2(t),-1\right).$$

Note that

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{2\gamma_1(t)}{\sqrt{4\left(\gamma_1(t)^2+\gamma_2(t)^2\right)+1}}\right)=\frac{2\gamma_1'(t)\sqrt{4\left(\gamma_1(t)^2+\gamma_2(t)^2\right)+1}-\frac{4\left(\gamma_1(t)\gamma_1'(t)+\gamma_2(t)\gamma_2'(t)\right)2\gamma_1(t)}{4\left(\gamma_1(t)^2+\gamma_2(t)^2\right)+1}}{4\left(\gamma_1(t)^2+\gamma_2(t)^2\right)+1}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2\gamma_1(t)}{\sqrt{4\left(\gamma_1(t)^2+\gamma_2(t)^2\right)+1}}\right)\Bigg|_{t=0}=2v_1.$$

Similarly, one shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{2\gamma_2(t)}{\sqrt{4\left(\gamma_1(t)^2+\gamma_2(t)^2\right)+1}}\right)\Bigg|_{t=0}=-2v_2$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{\sqrt{4\left(\gamma_1(t)^2+\gamma_2(t)^2\right)+1}}\right)\Bigg|_{t=0}=0,$$

so that

$$\mathrm{d}N_0(v_1,v_2,v_2)=(2v_1,-2v_2,0).$$

Then note that $X_1=(0,1,0), X_2=(1,0,0)\in T_0S$ *are unit vectors with*

$$\mathrm{d}N_0(X_1)=-2X_1$$

and

$$\mathrm{d}N_0(X_2)=2X_2.$$

Thus $\lambda_1(0)=-2$ *and* $\lambda_2(0)=2.$ *This implies that* $K(0)=-4$ *and* $H(0)=0.$

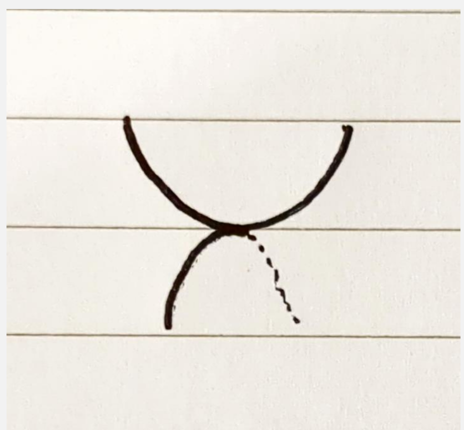


Figure 3.8.1: A surface with negative Gauss curvature.

Exercise 3.8.4. *Compute the Gauss curvature and mean curvature for the*

1. *helicoid, and*
2. *catenoid*

from Example 3.2.1.

Remark 3.8.5. *If the orientation of* S *is reversed, that is* N *is replaced with* $-N$*, then the sign of* H *changes whereas the sign of* K *does not.*

Proposition 3.8.6. *For a compact regular surface S, there exists a $p \in S$ such that $K(p) > 0$.*

Proof. There exists a point $p \in S$ that attains the maximum of the continuous map $x \mapsto |x|$. Thus, for any curve $\gamma : (\epsilon, \epsilon) \to S$ with $\gamma(0) = p$ by Proposition 2.3.10 we have that

$$|k_\gamma(0)| \geq \frac{1}{|\gamma(0)|} = \frac{1}{|p|}. \tag{3.8.1}$$

Recall, $k_\gamma(0)$ is equal to the normal curvature of the surface S at p in the direction $\gamma'(0)$. Thus, from Theorem 3.7.1 as $|\gamma'(0)| = 1$, we have that

$$|k_\gamma(0)| = |A_p(\gamma'(0), \gamma'(0))|.$$

Thus, using (3.8.1) it follows that for any $v \in T_pS$ with $|v| = 1$ we have

$$|A_p(v, v)| \geq \frac{1}{|p|}. \tag{3.8.2}$$

Now suppose $K(p) < 0$, then assuming $\lambda_1(p) \leq \lambda_2(p)$ it must be the case that $\lambda_1(p) < 0$ and $\lambda_2(p) > 0$. In particular, by continuity and using Proposition 3.6.8 it follows that there exists a $v \in T_pS$ such that $A_p(v, v) = 0$, which contradicts (3.8.2) as $\frac{1}{|p|} > 0$. Therefore, $\lambda_1(p)$ and $\lambda_2(p)$ have the same sign and thus $K(p) > 0$. ☐

Exercise 3.8.7. *Let $\gamma : I \to S$ be an asymptotic line as given by statement 1 of Remark 3.7.2.*

1. *Show that $K(p) \leq 0$ for all points $p \in \gamma(I)$.*
2. *Moreover, show that if $\mathbf{k}(t) \neq 0$ for all $t \in I$, then $|\tau(p)| = \sqrt{-K(p)}$, where τ is the torsion of γ.*

3.9 Umbilical Points

Definition 3.9.1. *Let S be an oriented surface.*

1. *A point $p \in S$ is referred to as an umbilical point if $\lambda_1(p) = \lambda_2(p)$.*
2. *The surface S is totally umbilic if every point $p \in S$ is umbilical.*

Remark 3.9.2. *At umbilical points, $\mathrm{d}N_p = -\lambda(p)\mathrm{id}$ and $A_p(x, y) = \lambda(p)\langle x, y\rangle$.*

Henceforth, we assume that all surfaces are connected.

Theorem 3.9.3 (Classification of Totally Umbilic Surfaces)**.** *Let S be an orientable surface that is totally umbilic. Then S is an open subset of either a plane or a sphere.*

Proof. Let $\phi : U \to S$ be a chart. There exists $\lambda : S \to \mathbb{R}$ such that $\lambda_1(p) = \lambda_2(p) = \lambda(p)$ for all $p \in S$.
Step 1: Show that λ is a constant function.
Observe that

$$\frac{\partial(N \circ \phi)}{\partial u}(q) = \mathrm{d}N_{\phi(q)}\left(\frac{\partial \phi}{\partial u}(q)\right) = -\lambda(\phi(q))\frac{\partial \phi}{\partial u}(q) \tag{3.9.1}$$

and

$$\frac{\partial(N \circ \phi)}{\partial v}(q) = \mathrm{d}N_{\phi(q)}\left(\frac{\partial \phi}{\partial v}(q)\right) = -\lambda(\phi(q))\frac{\partial \phi}{\partial v}(q) \tag{3.9.2}$$

for all $q \in U$. Hence,

$$\frac{\partial^2(N \circ \phi)}{\partial u \partial v} = -\partial_v(\lambda \circ \phi)\frac{\partial \phi}{\partial u} - (\lambda \circ \phi)\frac{\partial^2 \phi}{\partial u \partial v}$$

and

$$\frac{\partial^2 (N\circ\phi)}{\partial u \partial v} = -\partial_u(\lambda\circ\phi)\frac{\partial\phi}{\partial v} - (\lambda\circ\phi)\frac{\partial^2\phi}{\partial u\partial v}$$

where we have used the fact $N\circ\phi$ and ϕ are smooth and so the derivatives commutes. Thus,

$$\partial_u(\lambda\circ\phi)\frac{\partial\phi}{\partial v} = \partial_v(\lambda\circ\phi)\frac{\partial\phi}{\partial u}$$

in U. However, since $\left\{\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\right\}$ are linearly independent it follows that

$$\partial_u(\lambda\circ\phi) = \partial_v(\lambda\circ\phi) = 0,$$

that is $\lambda\circ\phi : U \to \mathbb{R}$ is constant. As ϕ is an arbitrary chart it follows that λ is constant.
Step 2: Show that $\lambda \equiv 0$ implies S is an open subset of a plane.
In this case, $\mathrm{d}N \equiv 0$ which implies that $N(p) = v$ for some $v \in \mathrm{S}^2$ and all $p \in S$. Let $p, p' \in \phi(U)$. Consider $\gamma : [0,1] \to U$ a smooth curve with $\gamma(0) = \phi^{-1}(p)$ and $\gamma(1) = \phi^{-1}(p')$. Then $\frac{\mathrm{d}(\phi\circ\gamma)}{\mathrm{d}t}(t) \in T_{\gamma(t)}S$ for all $t \in [0,1]$ and hence orthogonal to v. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi(\gamma(t)), v\rangle = \left\langle \frac{\mathrm{d}(\phi\circ\gamma)}{\mathrm{d}t}(t), v \right\rangle = 0.$$

In particular,

$$\langle p, v\rangle = \langle\phi(\gamma(0)), v\rangle = \langle\phi(\gamma(1)), v\rangle = \langle p', v\rangle.$$

As ϕ is arbitrary and S is connected it follows that

$$\langle p, v\rangle = \langle p', v\rangle$$

for all $p, p' \in S$, that is S is contained in a plane.
Step 3: Show that $\lambda \neq 0$ implies S is an open subset of a sphere.
Let us assume without loss of generality that $\lambda \equiv \lambda_0 > 0$. From (3.9.1) and (3.9.2) we obtain

$$\frac{\partial}{\partial u}\left(\phi + \frac{1}{\lambda_0}(N\circ\phi)\right) = 0$$

and

$$\frac{\partial}{\partial v}\left(\phi + \frac{1}{\lambda_0}(N\circ\phi)\right) = 0.$$

Thus, $\phi + \frac{1}{\lambda_0}(N\circ\phi)$ is constant on U and thus constant on S. Let $\phi + \frac{1}{\lambda_0}(N\circ\phi) \equiv v \in \mathbb{R}^3$, then

$$|\phi - v| = \left|\frac{1}{\lambda_0}(N\circ\phi)\right| = \frac{1}{\lambda_0},$$

which means that $\phi(U)$ is contained in the sphere of radius $\frac{1}{\lambda_0}$ with centre at v. As S is connected we deduce that all of S is contained in the sphere of radius $\frac{1}{\lambda_0}$ centered v. □

3.10 Elliptic, Parabolic and Hyperbolic Points

Exercise 3.10.1. *For an oriented surface S, let $\gamma : (-\epsilon, \epsilon) \to S$ be a smooth map. Then,*

$$A_{\gamma(t)}(\gamma'(t), \gamma'(t)) = \langle\gamma''(t), N(\gamma(t))\rangle.$$

Principal curvatures are indicative of the local shape of the surface, and thus provide a method to categorise the points of a surface. In particular, for a surface S let $\phi : U \to S$ be a chart with $0 \in U$ and $\phi(0) = p \in S$. Given $c, d \in \mathbb{R}$ let $\gamma : (-\epsilon, \epsilon) \to S$ be given by $\gamma(t) = \phi(ct, dt)$. Note that

$$\gamma'(t) = c\frac{\partial\phi}{\partial u}(ct, dt) + d\frac{\partial\phi}{\partial v}(ct, dt)$$

and

$$\gamma''(t) = c^2\frac{\partial^2\phi}{\partial u^2} + 2cd\frac{\partial^2\phi}{\partial u\partial v} + d^2\frac{\partial^2\phi}{\partial v^2}.$$

Hence, using Exercise 3.10.1 it follows that

$$A_p\left(c\frac{\partial\phi}{\partial u}(0) + d\frac{\partial\phi}{\partial v}(0), c\frac{\partial\phi}{\partial u}(0) + d\frac{\partial\phi}{\partial v}(0)\right) = \left\langle c^2\frac{\partial^2\phi}{\partial u^2}(0) + 2cd\frac{\partial^2\phi}{\partial u\partial v}(0) + d^2\frac{\partial^2\phi}{\partial v^2}(0), N(p)\right\rangle. \tag{3.10.1}$$

Suppose without loss of generality that $\phi(0) = 0 \in \mathbb{R}^3$ and

$$\frac{\partial\phi}{\partial u}(0) = (1,0,0)$$

and

$$\frac{\partial\phi}{\partial v}(0) = (0,1,0)$$

are the principal directions at zero.

Proposition 3.10.2. *Let $\phi : U \to S$ be a chart for S as above. Then as $u, v \to 0$ we have*

$$\phi(u,v) = \left(u, v, \frac{1}{2}\left(\lambda_1 u^2 + \lambda_1 v^2\right)\right) + R(u,v),$$

where λ_1, λ_2 are the principal curvatures at zero. In particular, $R(u,v)$ is a function such that

$$\lim_{u,v\to 0}\frac{R_3(u,v)}{u^2+v^2} = 0$$

and

$$\lim_{u,v\to 0}\frac{R_i(u,v)}{|u|+|v|} = 0$$

for $i = 1, 2$.

Proof. Using a Taylor expansion we have

$$\phi(u,v) = (u,v,0) + \frac{1}{2}\left(\frac{\partial^2\phi}{\partial u^2}(0)u^2 + 2\frac{\partial^2\phi}{\partial u\partial v}(0)uv + \frac{\partial^2\phi}{\partial v^2}(0)v^2\right) + \tilde{R}(u,v), \tag{3.10.2}$$

where

$$\lim_{u,v\to 0}\frac{\tilde{R}(u,v)}{u^2+v^2} = 0.$$

Note that

$$N(0,0,0) = \frac{\frac{\partial\phi}{\partial u}(0)\times\frac{\partial\phi}{\partial v}(0)}{\left|\frac{\partial\phi}{\partial u}(0)\times\frac{\partial\phi}{\partial v}(0)\right|} = (0,0,1)$$

Hence,

$$\begin{aligned}\left\langle\frac{1}{2}\left(\frac{\partial^2\phi}{\partial u^2}(0)u^2 + 2\frac{\partial^2\phi}{\partial u\partial v}(0)uv + \frac{\partial^2\phi}{\partial v^2}(0)v^2\right), (0,0,1)\right\rangle &\overset{(3.10.1)}{=} \frac{1}{2}A_0\left((u,v,0),(u,v,0)\right)\\ &= \frac{1}{2}A_0\left(uX_1 + vX_2, uX_1 + vX_2\right)\\ &= \frac{1}{2}\left(A_0(X_1,X_1)u^2 + A_0(X_2,X_2)v^2\right)\\ &= \frac{1}{2}\left(\lambda_1 u^2 + \lambda_2 v^2\right).\end{aligned}$$

Thus, (3.10.2) can be written as

$$\phi(u,v) = \left(u, v, \frac{1}{2}\left(\lambda_1 u^2 + \lambda_2 v^2\right)\right) + R(u,v)$$

where

$$\lim_{u,v\to 0} \frac{R_3(u,v)}{u^2+v^2} = 0$$

and

$$\lim_{u,v\to 0} \frac{R_i(u,v)}{|u|+|v|} = 0$$

for $i = 1, 2$. □

Remark 3.10.3. *Proposition 3.10.2 says that locally around zero, S is approximated by the graph of the function $f(x,y) = \frac{1}{2}\left(\lambda_1 x^2 + \lambda_2 y^2\right)$.*

Definition 3.10.4. *Let S be an oriented surface.*

1. *A point $p \in S$ is elliptic if $K(p) > 0$. That is, $\lambda_1(p)$ and $\lambda_2(p)$ have the same sign.*
2. *A point $p \in S$ is hyperbolic if $K(p) < 0$. That is, $\lambda_1(p)$ and $\lambda_2(p)$ have the opposite sign.*
3. *A point $p \in S$ is parabolic if $K(p) = 0$ and $H(p) \neq 0$. That is, exactly one of $\lambda_1(p)$ and $\lambda_2(p)$ is non-zero.*
4. *A point $p \in S$ is planar if $K(p) = H(p) = 0$. That is, $\lambda_1(p) = \lambda_2(p) = 0$.*

Example 3.10.5.

1. *Every point on S^2_r is elliptic.*
2. *Every point on C_r is parabolic.*
3. *Every point on $\left\{x^2 - y^2 = z\right\}$ is hyperbolic.*
4. *Every point on the plane is planar.*
5. *Every point on the paraboloid $\left\{z = x^2 + y^2\right\}$ is elliptic.*

Proposition 3.10.6. *Let $S \subseteq \mathbb{R}^3$ be a regular surface. For $p \in S$ the following statements hold.*

1. *If p is an elliptic point, then there is a neighbourhood $V \subseteq \mathbb{R}^3$ of p such that $S \cap V$ lies on one side of $p + T_pS$ and $(S \cap V) \cap (p + T_pS) = \{p\}$.*
2. *If p is a hyperbolic point, then on any neighbourhood $V \subseteq \mathbb{R}^3$ of p, $S \cap V$ meets both sides of $p + T_pS$.*

Proof. Let $\phi : U \to S$ be a chart for S at p, with $\phi(0,0) = p$. Then by Taylor's theorem

$$\begin{aligned}\phi(u,v) =& \phi(0,0) + \left(\frac{\partial\phi}{\partial u}(0,0)u + \frac{\partial\phi}{\partial v}(0,0)v\right) \\ &+ \frac{1}{2}\left(\frac{\partial^2\phi}{\partial u^2}(0,0)u^2 + 2\frac{\partial^2\phi}{\partial u\partial v}(0,0)uv + \frac{\partial^2\phi}{\partial v^2}(0,0)v^2\right) + R(u,v)\end{aligned}$$

in a neighbourhood of zero and where

$$\lim_{(u,v)\to(0,0)} \frac{R(u,v)}{u^2+v^2} = 0.$$

Since, $\frac{\partial \phi}{\partial u}(0,0)u + \frac{\partial \phi}{\partial v}(0,0)v \in T_pS$ it follows that

$$\begin{aligned}\langle \phi(u,v) - p, N(p)\rangle &= \frac{1}{2}\left\langle \frac{\partial^2\phi}{\partial u^2}(0,0)u^2 + 2\frac{\partial^2\phi}{\partial u \partial v}(0,0)uv + \frac{\partial^2\phi}{\partial v^2}(0,0)v^2, N(p)\right\rangle \\ &\quad + \langle R(u,v), N(p)\rangle \\ &\overset{(3.10.1)}{=} \frac{1}{2}A_p\left(\frac{\partial\phi}{\partial u}(0,0)u + \frac{\partial\phi}{\partial v}(0,0)v, \frac{\partial\phi}{\partial u}(0,0)u + \frac{\partial\phi}{\partial v}(0,0)v\right) \\ &\quad + \langle R(u,v), N(p)\rangle. \end{aligned} \tag{3.10.3}$$

- If $K(p) > 0$, then without loss of generality suppose that $\lambda_2(p) \geq \lambda_1(p) > 0$. Then by Proposition 3.6.8, it follows that for every $w \in T_pS \setminus \{0\}$ we have

  $$A_p(w,w) \geq \lambda_1(p)|w|^2 > 0.$$

  Thus, with (3.10.3) it follows that

  $$\langle \phi(u,v) - p, N(p)\rangle > 0$$

  for small values of u and v with $(u,v) \neq (0,0)$.

- If $K(p) < 0$ then $\lambda_1(p)$ and $\lambda_2(p)$ have different signs. Thus when w points in a principal direction we have that $A_p(w,w) = \lambda_1(p)|w|^2$ or $A_p(w,w) = \lambda_2(p)|w|^2$. Thus, A_p takes on both positive and negative values. Therefore, from (3.10.3) it follows that $\langle \phi(u,v) - p, N(p)\rangle$ takes positive and negative values in any neighbourhood of $(0,0)$.

$\square$

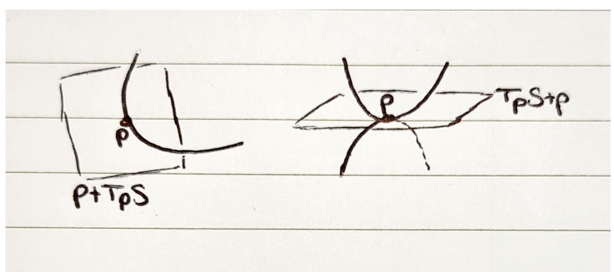


Figure 3.10.1: An illustration of the conclusions of Proposition 3.10.6.

3.11 Gauss and Mean Curvature in Charts

Definition 3.11.1. *Let S be an oriented surface with a chart $\phi : U \to S$.*

1. *Let $M := \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ where $E = \left|\frac{\partial\phi}{\partial u}\right|^2$, $F = \left\langle \frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\right\rangle$ and $G = \left|\frac{\partial\phi}{\partial v}\right|^2$.*

2. *Let $\Sigma := \begin{pmatrix} e & f \\ f & g \end{pmatrix}$ where $e = A_p\left(\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial u}\right) = \left\langle \frac{\partial^2\phi}{\partial u^2}, N(p)\right\rangle$, $f = A_p\left(\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\right) = \left\langle \frac{\partial^2\phi}{\partial u\partial v}, N(p)\right\rangle$ and $g = A_p\left(\frac{\partial\phi}{\partial v}, \frac{\partial\phi}{\partial v}\right) = \left\langle \frac{\partial^2\phi}{\partial v^2}, N(p)\right\rangle$.*

Remark 3.11.2. *Note that functions E, F, G, e, f and g of Definition 3.11.1 are functions $U \to \mathbb{R}$.*

Proposition 3.11.3. *With the notation of Definition 3.11.1 we have*

$$K(\phi(u,v)) = \det\left(M^{-1}\Sigma\right) = \frac{\det(\Sigma)}{\det(M)} = \frac{eg - f^2}{EG - F^2},$$

and

$$H(\phi(u,v)) = \frac{1}{2}\mathrm{tr}\left(M^{-1}\Sigma\right) = \frac{1}{2}\frac{eG + gE - 2fF}{EG - F^2}.$$

Proof. Fix an orthonormal basis for T_pS and represent $\mathrm{d}N\phi(u,v)$, ϕ_u and ϕ_v in the basis such that

$$\begin{aligned}
\Sigma &= \begin{pmatrix} - & \phi_u & - \\ - & \phi_v & - \end{pmatrix} \begin{pmatrix} | & | \\ -\mathrm{d}N_{\phi(u,v)}(\phi_u) & -\mathrm{d}N_{\phi(u,v)}(\phi_v) \\ | & | \end{pmatrix} \\
&= \begin{pmatrix} - & \phi_u & - \\ - & \phi_v & - \end{pmatrix} \left(-\mathrm{d}N_{\phi(u,v)}\right) \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}.
\end{aligned}$$

It follows that,

$$\begin{aligned}
\det(\Sigma) &= \det\left(\begin{pmatrix} - & \phi_u & - \\ - & \phi_v & - \end{pmatrix}\right) \det\left(-\mathrm{d}N_{\phi(u,v)}\right) \det\left(\begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}\right) \\
&= \lambda_1(p)\lambda_2(p) \det\left(\begin{pmatrix} - & \phi_u & - \\ - & \phi_v & - \end{pmatrix}\begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}\right) \\
&= K(\phi(u,v))\det(M).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Sigma &= \begin{pmatrix} - & \phi_u & - \\ - & \phi_v & - \end{pmatrix} \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix} \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}^{-1} \left(-\mathrm{d}N_{\phi(u,v)}\right) \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix} \\
&= M\left(\begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}^{-1} \left(-\mathrm{d}N_{\phi(u,v)}\right) \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}\right).
\end{aligned}$$

So that,

$$\frac{1}{2}\mathrm{tr}\left(M^{-1}\Sigma\right) = \frac{1}{2}\mathrm{tr}\left(-\mathrm{d}N_{\phi(u,v)}\right) = \frac{\lambda_1(p) + \lambda_2(p)}{2} = H(p).$$

□

Remark 3.11.4. *When* $\frac{\partial\phi}{\partial u}$ *and* $\frac{\partial\phi}{\partial v}$ *are the principal directions we have* $M = I$ *and* $\Sigma = \begin{pmatrix} \lambda_1(p) & 0 \\ 0 & \lambda_2(p) \end{pmatrix}$. *Therefore,*

$$K(p) = \det(\Sigma) = \lambda_1(p)\lambda_2(p)$$

and

$$H(p) = \frac{1}{2}\mathrm{tr}\left(\Sigma\right) = \frac{\lambda_1(p) + \lambda_2(p)}{2}$$

as expected.

Example 3.11.5. *Let* $S = \left\{z = x^2 - y^2\right\}$*. Then a chart for* S *is* $\phi : \mathbb{R}^2 \to S$ *given by*

$$\phi(u,v) = \left(u, v, u^2 - v^2\right).$$

Note that

$$\frac{\partial \phi}{\partial u} = (1, 0, 2u)$$

and

$$\frac{\partial \phi}{\partial v} = (0, 1, -2v)$$

so that

$$N(\phi(u,v)) = \frac{\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}}{\left|\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}\right|} = \frac{(-2u, 2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}}.$$

Moreover,

$$\begin{cases} \frac{\partial^2 \phi}{\partial u^2} = (0,0,2) \\ \frac{\partial^2 \phi}{\partial u \partial v} = (0,0,0) \\ \frac{\partial^2 \phi}{\partial v^2} = (0,0,-2) \end{cases}$$

so that

$$M = \begin{pmatrix} 1 + 4u^2 & -4uv \\ -4uv & 1 + 4v^2 \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} \frac{2}{\sqrt{1+4u^2+4v^2}} & 0 \\ 0 & -\frac{2}{\sqrt{1+4u^2+4v^2}} \end{pmatrix}.$$

Therefore,

$$\begin{aligned} K(\phi(u,v)) &= \det\left(M^{-1}\Sigma\right) \\ &= \frac{-\frac{4}{1+4u^2+4v^2}}{\left(1+4u^2\right)\left(1+4v^2\right) - 16u^2v^2} \\ &= -\frac{4}{\left(1+4u^2+4v^2\right)^2} \end{aligned}$$

and

$$\begin{aligned} H(\phi(u,v)) &= \frac{1}{2}\mathrm{tr}\left(\frac{1}{1+4u^2+v^2}\begin{pmatrix} 1+4v^2 & 4uv \\ 4uv & 1+4u^2 \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{1+4u^2+4v^2}} & 0 \\ 0 & -\frac{2}{\sqrt{1+4u^2+4v^2}} \end{pmatrix}\right) \\ &= \frac{1}{2\left(1+4u^2+4v^2\right)}\mathrm{tr}\begin{pmatrix} \frac{2\left(1+4v^2\right)}{\sqrt{1+4u^2+4v^2}} & -\frac{8uv}{\sqrt{1+4u^2+4v^2}} \\ \frac{8uv}{\sqrt{1+4u^2+4v^2}} & -\frac{2\left(1+4u^2\right)}{\sqrt{1+4u^2+4v^2}} \end{pmatrix} \\ &= \frac{4v^2 - 4u^2}{\left(1+4u^2+4v^2\right)^{\frac{3}{2}}}. \end{aligned}$$

Exercise 3.11.6. *Repeat Exercise 3.8.4 by using Proposition 3.11.3.*

Example 3.11.7. *Using the computations of Exercise 3.11.6 one can identify the asymptotic lines of the helicoid and catenoid.*

1. *For the helicoid S, consider the chart $\phi : U \to S$ given by*

$$\phi(u, v) = (u\cos(v), u\sin(v), v).$$

Then a regular curve $\gamma : I \to S$ can be written as $\gamma(t) = (\phi \circ \sigma)(t)$ where $\sigma : I \to U$ is given by $\sigma(t) = (u(t), v(t))$ for smooth curves u, v. In particular,

$$\gamma'(t) = \frac{\partial \phi}{\partial u} u'(t) + \frac{\partial \phi}{\partial v} v'(t).$$

Therefore,

$$A_{\gamma(t)}\left(\gamma'(t), \gamma'(t)\right) = \begin{pmatrix} u'(t) & v'(t) \end{pmatrix} \Sigma \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}$$

where

$$\Sigma = \frac{1}{\sqrt{u(t)^2 + 1}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

is the matrix of $A_{\gamma(t)}$ with respect to the basis $\left\{ \frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial t} \right\}$ as determined in Exercise 3.11.6. Therefore, γ is an asymptotic line if and only if

$$-\frac{2u'(t)v'(t)}{\sqrt{u(t)^2 + 1}} = 0$$

for all $t \in I$. That is, either $u'(t) = 0$ or $v'(t) = 0$. Therefore, the asymptotic lines of the helicoid are

$$\gamma(t) = (c\cos(v(t)), c\sin(v(t)), v(t))$$

and

$$\gamma(t) = (u(t)\cos(c), u(t)\sin(c), c)$$

for $c \in \mathbb{R}$.

2. *For the catenoid S, consider the chart $\phi : U \to S$ given by*

$$\phi(u, v) = (\cosh(u)\cos(v), \cosh(u)\sin(v), u).$$

Then a regular curve $\gamma : I \to S$ can be written as $\gamma(t) = (\phi \circ \sigma)(t)$ where $\sigma : I \to U$ is given by $\sigma(t) = (u(t), v(t))$ for smooth curves u, v. In particular,

$$\gamma'(t) = \frac{\partial \phi}{\partial u} u'(t) + \frac{\partial \phi}{\partial v} v'(t).$$

Therefore,

$$A_{\gamma(t)}\left(\gamma'(t), \gamma'(t)\right) = \begin{pmatrix} u'(t) & v'(t) \end{pmatrix} \Sigma \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}$$

where

$$\Sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the matrix of $A_{\gamma(t)}$ with respect to the basis $\left\{ \frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial t} \right\}$ as determined in Exercise 3.11.6. Therefore, γ is an asymptotic line if and only if

$$-\left(u'(t)\right)^2 + \left(v'(t)\right)^2 = 0$$

for all $t \in I$. That is, either $u'(t) - v'(t) = 0$ or $u'(t) + v'(t) = 0$. Therefore, the asymptotic lines of the catenoid are

$$\gamma(t) = (\cosh(u(t))\cos(u(t)+c), \cosh(u(t))\sin(u(t)+c), u(t))$$

and

$$\gamma(t) = (\cosh(u(t))\cos(c-u(t)), \cosh(u(t))\sin(c-u(t)), u(t))$$

for $c \in \mathbb{R}$.

3.12 First Fundamental Form

The intrinsic geometry of a surface is captured by the first fundamental form.

Definition 3.12.1. *The first fundamental form of a surface S at $p \in S$ is the bilinear map $g : T_pS \times T_pS \to \mathbb{R}$ given by $g(x, y) = \langle x, y\rangle$.*

Remark 3.12.2. *The first fundamental form at a point p is the restriction of the Euclidean dot product onto the tangent space of S at p.*

If $\phi : U \to S$ is a chart for S, then $T_pS = \operatorname{span}\left(\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\right)$. In particular,

$$\begin{aligned}
g(x,y) &= g\left(a\frac{\partial\phi}{\partial u} + b\frac{\partial\phi}{\partial v}, c\frac{\partial\phi}{\partial u} + d\frac{\partial\phi}{\partial v}\right) \\
&= ac\left\langle\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial u}\right\rangle + (ad+bc)\left\langle\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\right\rangle + bd\left\langle\frac{\partial\phi}{\partial v}, \frac{\partial\phi}{\partial v}\right\rangle \\
&= \begin{pmatrix} a & b\end{pmatrix} M \begin{pmatrix} c \\ d\end{pmatrix}.
\end{aligned}$$

That is, M is the matrix of g with respect to the basis $\left\{\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\right\}$. In particular, this allows us to write intrinsic notions of curves such as length using the first fundamental form. More specifically, let $\gamma : (a,b) \to S$ be a regular curve and suppose that $\gamma((a,b)) \subseteq \phi(U)$. Recall that

$$L(\gamma) = \int_a^b \left|\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right| \mathrm{d}t.$$

Then supposing that $\gamma(t) = \phi(u(t), v(t))$ it follows that

$$\begin{aligned}
L(\gamma) &= \int_a^b \left|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\frac{\partial\phi}{\partial u}(u(t),v(t)) + \frac{\mathrm{d}v}{\mathrm{d}t}(t)\frac{\partial\phi}{\partial v}(u(t),v(t))\right| \mathrm{d}t \\
&= \int_a^b \sqrt{\begin{pmatrix}\frac{\mathrm{d}u}{\mathrm{d}t} & \frac{\mathrm{d}v}{\mathrm{d}t}\end{pmatrix} M \begin{pmatrix}\frac{\mathrm{d}u}{\mathrm{d}t} \\ \frac{\mathrm{d}v}{\mathrm{d}t}\end{pmatrix}}\, \mathrm{d}t \\
&= \int_a^b \sqrt{\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 E + 2\frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}t}F + \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 G}\, \mathrm{d}t.
\end{aligned}$$

Similarly, one can express angles using the first fundamental form.

Definition 3.12.3. *Let $S_1, S_2 \subseteq \mathbb{R}^3$ be regular surfaces. A smooth map $F : S_1 \to S_2$ is a local isometry if it preserves the first fundamental form. That is, for any $p \in S_1$ we have*

$$\langle \mathrm{d}F_p(x), \mathrm{d}F_p(y)\rangle = \langle x, y\rangle$$

for all $x, y \in T_pS_1$. Moreover, F is an isometry if it is a local isometry and a bijection between S_1 and S_2.

Remark 3.12.4. *Let $F : S_1 \to S_2$ be a local isometry.*

1. *Then $\mathrm{d}F_p : T_pS_1 \to T_{F(p)}S_2$ is a bijection for all $p \in S_1$. Indeed, if $\mathrm{d}F_p(v) = 0$ then*

$$|v|^2 = \langle v, v\rangle = \langle \mathrm{d}F_p(v), \mathrm{d}F_p(v)\rangle = |\mathrm{d}F_p(v)|^2 = 0$$

and so $v = 0$. Therefore, since T_pS_1 and $T_{F(p)}S_2$ are finite-dimensional linear spaces it follows that $\mathrm{d}F_p$ is surjective.

2. *From Theorem 3.2.5 there are neighbourhoods $U_1 \subseteq S_1$ of p and $U_2 \subseteq S_2$ of $F(p)$ such that $F|_{U_1} : U_1 \to U_2$ is a diffeomorphism.*

Example 3.12.5.

1. *Rigid motions of $\mathbb{R}^3$ induce isometries. Let $S \subseteq \mathbb{R}^3$, then for $A \in \mathrm{SO}(3)$ we can consider*

$$S' = A(S) = \{Ap : p \in S\}.$$

Then let $F : S \to S'$ be given by $p \mapsto Ap$. Observe that F is smooth. Moreover, for any $p \in S$ let $v \in T_pS$. Then for $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^3$ with $\gamma'(0) = v$ we have

$$\mathrm{d}F_p(v) = \frac{\mathrm{d}(A\gamma(t))}{\mathrm{d}t}\Big|_{t=0} = A\gamma'(t)|_{t=0} = Av.$$

Therefore,

$$\begin{aligned}\langle \mathrm{d}F_p(v), \mathrm{d}F_p(w)\rangle &= \langle Av, Aw\rangle \\ &= (Av)^\top(Aw) \\ &= v^\top A^\top Aw \\ &= v^\top w \\ &= \langle v, w\rangle\end{aligned}$$

which means that F is a local isometry. As A is invertible, F is an isometry.

2. *Let $S_1 = \big\{(x, y, z) \in \mathbb{R}^3 : z = 0\big\}$ and $S_2 = \big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\big\}$. Let $F : S_1 \to S_2$ be given by*

$$F(x, y, 0) = (\cos(x), \sin(x), y).$$

Then for any $(x, y, 0) \in S_1$, the tangent plane is S_1, namely $T_pS_1 = S_1$. In particular, for any $v = (v_1, v_2, 0) \in T_pS_1$ write $\gamma(t) = (x + tv_1, y + tv_2, 0)$ so that $\gamma(0) = (x, y, 0)$ and $\gamma'(0) = v$. Then,

$$\begin{aligned}\mathrm{d}F_{(x,y,0)}(v) &= \frac{\mathrm{d}F(\gamma(t))}{\mathrm{d}t}\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\cos(x + tv_1), \sin(x + tv_1), y + tv_2\right)|_{t=0} \\ &= (-v_1\sin(x), v_1\cos(x), v_2).\end{aligned}$$

Hence,

$$\begin{aligned}\left\langle \mathrm{d}F_{(x,y,0)}(v), \mathrm{d}F_{(x,y,0)}(v)\right\rangle &= (v_1)^2\left(\sin^2(x)+\cos^2(x)\right)+(v_2)^2\\ &= (v_1)^2+(v_2)^2\\ &= \langle v, v\rangle.\end{aligned}$$

It follows that

$$\left\langle \mathrm{d}F_{(x,y,0)}(x), \mathrm{d}F_{(x,y,0)}(y)\right\rangle = \langle x, y\rangle$$

for all $x, y \in T_pS_1$. Indeed,

$$\begin{aligned}\left\langle \mathrm{d}F_{(x,y,0)}(x), \mathrm{d}F_{(x,y,0)}(y)\right\rangle &= \frac{1}{2}\left(\left|\mathrm{d}F_{(x,y,0)}(x+y)\right|^2-\left|\mathrm{d}F_{(x,y,0)}(x)\right|^2-\left|\mathrm{d}F_{(x,y,0)}(y)\right|^2\right)\\ &= \frac{1}{2}\left(|x+y|^2-|x|^2-|y|^2\right)\\ &= \langle x, y\rangle.\end{aligned}$$

Thus, F is a local isometry, however, it is not an isometry as F is not injective.

Lemma 3.12.6. *Let $\varphi : U \to S$ and $\tilde{\varphi} : U \to \tilde{S}$ be charts. Suppose that*

$$M = \begin{pmatrix}|\varphi_u|^2 & \langle\varphi_u, \varphi_v\rangle\\ \langle\varphi_v, \varphi_u\rangle & |\varphi_v|^2\end{pmatrix} = \begin{pmatrix}|\tilde{\varphi}_u|^2 & \langle\tilde{\varphi}_u, \tilde{\varphi}_v\rangle\\ \langle\tilde{\varphi}_v, \tilde{\varphi}_u\rangle & |\tilde{\varphi}_v|^2\end{pmatrix} = \tilde{M}.$$

Then $f : \varphi(U) \to \tilde{S}$ given by $f = \tilde{\varphi}\circ\varphi^{-1}$ is a local isometry.

Proof. Let $p = \varphi(q)$ where $q = (q_1, q_2)$. Then

$$\begin{aligned}\mathrm{d}f_p(\varphi_u(q)) &= \frac{\mathrm{d}}{\mathrm{d}t}f\left(\varphi(q_1+t, q_2)\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\varphi}(q_1+t, q_2)\\ &= \tilde{\varphi}_u(q).\end{aligned}$$

Similarly, $\mathrm{d}f_p\left(\varphi_v(q)\right) = \tilde{\varphi}_v(q)$. Therefore, for $v \in T_pS$ we have

$$\begin{aligned}\langle \mathrm{d}f_p(v), \mathrm{d}f_p(v)\rangle &= \langle \mathrm{d}f_p(a\varphi_u(q)+b\varphi_v(q)), \mathrm{d}f_p(a\varphi_u(q)+b\varphi_v(q))\rangle\\ &= \langle a\tilde{\varphi}_u(q)+b\tilde{\varphi}_v(q), a\tilde{\varphi}_u(q)+b\tilde{\varphi}_v(q)\rangle\\ &= \begin{pmatrix}a & b\end{pmatrix}\tilde{M}\begin{pmatrix}a\\ b\end{pmatrix}\\ &= \begin{pmatrix}a & b\end{pmatrix}M\begin{pmatrix}a\\ b\end{pmatrix}\\ &= \langle a\varphi_u(q)+b\varphi_v(q), a\varphi_u(q)+b\varphi_v(q)\rangle\\ &= \langle v, v\rangle.\end{aligned}$$

Therefore, f is a local isometry. ☐

Exercise 3.12.7. *Let*

$$S_1 = \left\{(\cosh(u)\cos(v), \cosh(u)\sin(v), u) : (u, v)\in\mathbb{R}^2\right\}$$

be the catenoid and let

$$S_2 = \left\{(u\cos(v), u\sin(v), v) : (u, v)\in\mathbb{R}^2\right\}$$

be the helicoid. Show that S_1 and S_2 are locally isometric.

Proposition 3.12.8. *Let $F : S_1 \to S_2$ be a smooth map between regular surfaces. Then F is a local isometry if and only if F preserves the length of all curves. That is, for any smooth curve $\gamma : (a,b) \to S_1$ we have*

$$L_{S_1}(\gamma) = L_{S_2}(F \circ \gamma).$$

Proof. $(\Rightarrow)$. Using $(F\circ\gamma)'(t) = \mathrm{d}F_{\gamma(t)}\left(\gamma'(t)\right)$ it follows that

$$\begin{aligned} L_{S_2}(F\circ\gamma) &= \int_a^b |(F\circ\gamma)'(t)| \ \mathrm{d}t \\ &= \int_a^b \left|\mathrm{d}F_{\gamma(t)}\left(\gamma'(t)\right)\right| \ \mathrm{d}t \\ &= \int_a^b \sqrt{\left\langle \mathrm{d}F_{\gamma(t)}\left(\gamma'(t)\right), \mathrm{d}F_{\gamma(t)}\left(\gamma'(t)\right)\right\rangle}\, \mathrm{d}t \\ &\overset{(1)}{=} \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t)\rangle}\, \mathrm{d}t \\ &= \int_a^b |\gamma'(t)| \ \mathrm{d}t \\ &= L_{S_1}(\gamma), \end{aligned}$$

where in (1) the local isometry property of F is used.
$(\Leftarrow)$. Let $p \in S_1$ and consider $v \in T_pS_1$. Choose $\gamma : (-\epsilon, \epsilon) \to S_1$ a smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then for all $0 < t < \epsilon$ we have

$$L_{S_1}\left(\gamma|_{(-\epsilon,t)}\right) = L_{S_2}\left((F\circ\gamma)|_{(-\epsilon,t)}\right).$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\epsilon}^t |\gamma'(s)| \ \mathrm{d}s\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\epsilon}^t |(F\circ\gamma)'(s)| \ \mathrm{d}s\Big|_{t=0}.$$

Therefore, $|\gamma'(0)| = |(F\circ\gamma)'(0)|$, or in other words, $|v| = |\mathrm{d}F_p(v)|$. As p and v were arbitrary it follows that

$$|v| = |\mathrm{d}F_p(v)|$$

for all $p \in S_1$ and $v \in T_pS_1$ which means that F is a local isometry using the same reasoning as that used in statement 2 of Example 3.12.5. ☐

Example 3.12.9. *Consider the regular surfaces*

$$S_1 = \{\phi(u,v) = (u\cos(v), u\sin(v), \log(u)) : u > 0, v \in \mathbb{R}\}$$

and

$$S_2 = \{\psi(u,v) = (u\cos(v), u\sin(v), v) : u > 0, v \in \mathbb{R}\}.$$

One can show that the Gauss curvatures of S_1 and S_2 are equal, namely

$$K_{S_1}(\phi(u,v)) = K_{S_2}(\psi(u,v)) = -\frac{1}{u^2+1}.$$

Consider $F : S_1 \to S_2$ given by $F = \psi \circ \phi^{-1}$. Then,

$$\langle \partial_u\phi, \partial_u\phi\rangle = 1 + \frac{1}{u^2}$$

whereas

$$\left\langle \mathrm{d}F_{\phi(u,v)}\left(\partial_u\phi\right), \mathrm{d}F_{\phi(u,v)}\left(\partial_u\phi\right)\right\rangle = \langle\partial_u\psi, \partial_u\psi\rangle = 1.$$

Therefore, F is not a local isometry.

3.13 Christoffel Symbols

Let $\phi : U \to S$ be a chart of a regular surface S with Gauss map $N : S \to \mathrm{S}^2$. Then $\left\{\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}, N\right\}$ forms an orthonormal basis for $\mathbb{R}^3$. With $x^1 := u$ and $x^2 := v$, let

$$g_{ij} = g\left(\frac{\partial\phi}{\partial x^i}, \frac{\partial\phi}{\partial x^j}\right) = \left\langle \frac{\partial\phi}{\partial x^i}, \frac{\partial\phi}{\partial x^j}\right\rangle$$

for $i, j = 1, 2$. Moreover, let $\left(g^{ij}\right)$ be the inverse of the matrix $M = (g_{ij})$. That is,

$$\sum_{k=1}^{2} g_{ik}g^{kj} = \delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Note that $\frac{\partial^2\phi}{\partial x^i\partial x^j}$ can be written as

$$\frac{\partial^2\phi}{\partial x^i\partial x^j} = \Gamma^1_{ij}\frac{\partial\phi}{\partial x^1} + \Gamma^2_{ij}\frac{\partial\phi}{\partial x^2} + A_{ij}N \tag{3.13.1}$$

for some functions $\Gamma^k_{ij} : U \to \mathbb{R}$ with $i, j, k = 1, 2$, and where

$$\begin{aligned} A_{ij} &= \left\langle \frac{\partial^2\phi}{\partial x^i\partial x^j}, N\right\rangle \\ &= A\left(\frac{\partial\phi}{\partial x^i}, \frac{\partial\phi}{\partial x^j}\right). \end{aligned}$$

Definition 3.13.1. *The functions $\Gamma^k_{ij} : U \to \mathbb{R}$ for $i, j, k = 1, 2$ as determined by* (3.13.1) *are referred to as the Christoffel symbols of the chart ϕ.*

Remark 3.13.2. *As $\frac{\partial^2\phi}{\partial x^i\partial x^j} = \frac{\partial^2\phi}{\partial x^j\partial x^i}$ for $i, j = 1, 2$ it follows that $\Gamma^k_{ij} = \Gamma^k_{ji}$ for $i, j = 1, 2$.*

Proposition 3.13.3. *Let $\phi : U \to S$ be a chart for a regular surface S. Then*

$$\Gamma^k_{ij} = \sum_{l=1}^{2} \frac{g^{kl}}{2}\left(\partial_{x^i}g_{jl} + \partial_{x^j}g_{il} - \partial_{x^l}g_{ij}\right)$$

for $i, j, k = 1, 2$.

Proof. Using (3.13.1) we have

$$\begin{aligned} \left\langle \frac{\partial^2\phi}{\partial x^i\partial x^j}, \frac{\partial\phi}{\partial x^l}\right\rangle &= \Gamma^1_{ij}\left\langle \frac{\partial\phi}{\partial x^1}, \frac{\partial\phi}{\partial x^l}\right\rangle + \Gamma^2_{ij}\left\langle \frac{\partial\phi}{\partial x^2}, \frac{\partial\phi}{\partial x^l}\right\rangle + A_{ij}\left\langle N, \frac{\partial\phi}{\partial x^l}\right\rangle \\ &= \Gamma^1_{ij}g_{1l} + \Gamma^2_{ij}g_{2l}, \end{aligned}$$

for $l = 1, 2$. Hence,

$$\begin{aligned}
\partial_{x^i} g_{jl} &= \frac{\partial}{\partial x^i}\left\langle \frac{\partial \phi}{\partial x^j}, \frac{\partial \phi}{\partial x^l} \right\rangle \\
&= \left\langle \frac{\partial^2 \phi}{\partial x^i \partial x^j}, \frac{\partial \phi}{\partial x^l} \right\rangle + \left\langle \frac{\partial \phi}{\partial x^j}, \frac{\partial^2 \phi}{\partial x^i \partial x^l} \right\rangle \\
&= \sum_{m=1}^{2} \left(\Gamma_{ij}^m g_{ml} + \Gamma_{il}^m g_{jm} \right). \qquad (3.13.2)
\end{aligned}$$

Similarly,

$$\partial_{x^j} g_{il} = \sum_{m=1}^{2} \left(\Gamma_{ij}^m g_{ml} + \Gamma_{jl}^m g_{im} \right). \qquad (3.13.3)$$

and

$$\partial_{x^l} g_{ij} = \sum_{m=1}^{2} \left(\Gamma_{il}^m g_{mj} + \Gamma_{jl}^m g_{im} \right). \qquad (3.13.4)$$

Considering (3.13.2)+(3.13.3)-(3.13.4) it follows that

$$\partial_{x^i} g_{il} + \partial_{x^j} g_{il} - \partial_{x^l} g_{ij} = \sum_{m=1}^{2} 2\Gamma_{ij}^m g_{ml}.$$

Multiplying by $\frac{1}{2} g^{lk}$ and considering $l = 1, 2$, we conclude that

$$\begin{aligned}
\sum_{l=1}^{2} \frac{1}{2} g^{kl} \left(\partial_{x^i} g_{jl} + \partial_{x^j} g_{il} - \partial_{x^l} g_{ij} \right) &= \sum_{m=1}^{2} \sum_{l=1}^{2} \Gamma_{ij}^m g_{ml} g^{lk} \\
&= \sum_{m=1}^{2} \Gamma_{ij}^m \delta_m^k \\
&= \Gamma_{ij}^k.
\end{aligned}$$

□

Remark 3.13.4. *Let $F : S \to \tilde{S}$ be a local isometry. By shrinking U if necessary, the map $F \circ \phi : U \to \tilde{S}$ is a chart for $\tilde{S}$. In particular,*

$$\begin{aligned}
\tilde{g}_{ij} &= \left\langle \frac{\partial (F \circ \phi)}{\partial x^i}, \frac{\partial (F \circ \phi)}{\partial x^j} \right\rangle \\
&= \left\langle \mathrm{d}F\left(\frac{\partial \phi}{\partial x^i} \right), \mathrm{d}F\left(\frac{\partial \phi}{\partial x^j} \right) \right\rangle \\
&= \left\langle \frac{\partial \phi}{\partial x^i}, \frac{\partial \phi}{\partial x^j} \right\rangle \\
&= g_{ij}.
\end{aligned}$$

Thus, from Proposition 3.13.3 we have that

$$\tilde{\Gamma}_{ij}^k(u, v) = \Gamma_{ij}^k(u, v)$$

for $i, j, k = 1, 2$ and for all $(u, v) \in U$. In other words, Christoffel symbols are an intrinsic property of a surface. Consequently, any object expressible in terms of the first fundamental form is said to be an intrinsic property.

Example 3.13.5.

1. *Let $S_1 = \{z = 0\}$. Then $\phi : \mathbb{R}^2 \to S_1$ given by $\phi(u, v) = (u, v, 0)$ is a chart. In particular,*

$$\frac{\partial^2 \phi}{\partial u^2} = \frac{\partial^2 \phi}{\partial v^2} = \frac{\partial^2 \phi}{\partial u \partial v} = 0,$$

so

$$\Gamma_{ij}^k = 0$$

for $i, j, k = 1, 2$.

2. *Let $S_2 = \left\{x^2 + y^2 = 1\right\}$. Then $F : S_1 \to S_2$ given by $F(x, y, 0) = (\cos(x), \sin(x), y)$ was shown to be a local isometry in statement 2 of Example 3.12.5. For the chart $\psi(u, v) := (F \circ \phi)(u, v)$ of S_2 the Christoffel symbols of S_2 vanish as expected by Proposition 3.13.3. Indeed,*

$$\begin{cases} \frac{\partial \psi}{\partial u} = (-\sin(u), \cos(u), 0) \\ \frac{\partial \psi}{\partial v} = (0, 0, 1) \\ N = (\cos(u), \sin(u), 0) \end{cases}$$

so that

$$\begin{cases} \frac{\partial^2 \psi}{\partial u^2} = (-\cos(u), -\sin(u), 0) = -N \\ \frac{\partial^2 \psi}{\partial u \partial v} = (0, 0, 0) \\ \frac{\partial^2 \psi}{\partial v^2} = (0, 0, 0). \end{cases}$$

Exercise 3.13.6. *Let S be the graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ with chart $\phi : \mathbb{R}^2 \to S$ given by*

$$\phi(u, v) = (u, v, f(u, v)).$$

Compute the Christoffel symbols Γ_{ij}^k.

3.14 Theorem Egregium

Theorem 3.14.1 (Egregium)**.** *The Gauss curvature K of a regular surface S is intrinsic. More specifically, with $S, \tilde{S}$ and F as in Remark 3.13.4, it follows that*

$$K(\phi(u, v)) = \tilde{K}\left(\tilde{\phi}(u, v)\right),$$

where $\tilde{K}$ is the Gauss curvature of $\tilde{S}$. Indeed,

$$K(\phi(u, v)) = \frac{Y(u, v)G(u, v) - X(u, v)F(u, v)}{\det(M(u, v))} \tag{3.14.1}$$

where

$$X = \Gamma_{12}^2 \Gamma_{12}^1 - \Gamma_{11}^2 \Gamma_{22}^1 + \partial_u\left(\Gamma_{12}^1\right) - \partial_v\left(\Gamma_{11}^1\right)$$

and

$$Y = \Gamma_{11}^1 \Gamma_{21}^2 + \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{12}^1 \Gamma_{11}^2 - \left(\Gamma_{12}^2\right)^2 + \partial_v\left(\Gamma_{11}^2\right) - \partial_u\left(\Gamma_{12}^2\right).$$

Proof. It suffices to show (3.14.1) as M and the Christoffel symbols, Proposition 3.13.3, are invariant under local isometries. Let $\phi : U \to S$ be a chart for S. Since ϕ is smooth its partial derivatives commute, in particular,

$$\frac{\partial}{\partial v}\left(\frac{\partial^2 \phi}{\partial u \partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial^2 \phi}{\partial u \partial v}\right),$$

which can equivalently be written as

$$\frac{\partial}{\partial v}\left(\Gamma_{11}^{1}\frac{\partial \phi}{\partial u}+\Gamma_{11}^{2}\frac{\partial \phi}{\partial v}+A_{11}N\right)=\frac{\partial}{\partial u}\left(\Gamma_{12}^{1}\frac{\partial \phi}{\partial u}+\Gamma_{12}^{2}\frac{\partial \phi}{\partial v}+A_{12}N\right),$$

where N is the Gauss map. Thus,

$$\begin{aligned}\left(\Gamma_{11}^{1}\phi_{uv}+\left(\Gamma_{11}^{1}\right)_{v}\phi_{u}\right)+\left(\Gamma_{11}^{2}\phi_{vv}+\left(\Gamma_{11}^{2}\right)_{v}\phi_{v}\right)+\left(\left(A_{11}\right)_{v}N+A_{11}N_{v}\right)&=\left(\Gamma_{12}^{1}\phi_{uu}+\left(\Gamma_{12}^{1}\right)_{u}\phi_{u}\right)\\ &+\left(\Gamma_{12}^{2}\phi_{vu}+\left(\Gamma_{11}^{2}\right)_{u}\phi_{v}\right)+\left(\left(A_{12}\right)_{u}N+A_{12}N_{u}\right).\end{aligned}\tag{3.14.2}$$

As $\{\phi_u,\phi_v,N\}$ is linearly independent we can write the vectors ϕ_{uv}, ϕ_{vv}, ϕ_{uu}, N_u and N_v in terms of these vectors. Indeed, we know that

$$\begin{cases}\phi_{uv}=\frac{\partial^2\phi}{\partial v\partial u}=\Gamma_{11}^{1}\phi_u+\Gamma_{21}^{2}\phi_v+A_{12}N\\ \phi_{vv}=\frac{\partial^2\phi}{\partial v^2}=\Gamma_{22}^{1}\phi_u+\Gamma_{22}^{2}\phi_v+A_{22}N\\ \phi_{uu}=\frac{\partial^2\phi}{\partial u^2}=\Gamma_{11}^{1}\phi_u+\Gamma_{11}^{2}\phi_v+A_{11}N.\end{cases}\tag{3.14.3}$$

On the other hand, we know that

$$\begin{cases}N_u=a\phi_u+b\phi_v\\ N_v=c\phi_u+d\phi_v\end{cases}$$

for some $a,b,c,d\in\mathbb{R}$ as each is orthogonal to N. Then

$$\begin{aligned}-A_{11}&=-A(\phi_u,\phi_u)\\ &=\langle\phi_u,\mathrm{d}N(\phi_u)\rangle\\ &=\left\langle\phi_u,\frac{\partial N\circ\phi}{\partial u}\right\rangle\\ &=\langle\phi_u,N_u\rangle\\ &=aE+bF.\end{aligned}$$

Recalling that $\Sigma=(A_{ij})$, through similar computations we arrive at the matrix equation

$$-\Sigma=M\begin{pmatrix}a&c\\ b&d\end{pmatrix}.$$

Therefore,

$$\begin{pmatrix}a&c\\ b&d\end{pmatrix}=-\frac{1}{\det(M)}\begin{pmatrix}G&-F\\ -F&E\end{pmatrix}\begin{pmatrix}A_{11}&A_{12}\\ A_{21}&A_{22}\end{pmatrix}=-\frac{1}{\det(M)}M^{-1}\Sigma.$$

In particular, substituting the values of a and c into (3.14.2) along with (3.14.3) and comparing the coefficients of ϕ_u it follows that

$$\begin{aligned}\Gamma_{11}^{1}\Gamma_{21}^{1}+\left(\Gamma_{11}^{1}\right)_{v}+\Gamma_{11}^{2}\Gamma_{22}^{1}+A_{11}\left(\frac{FA_{22}-GA_{12}}{\det(M)}\right)&=\Gamma_{12}^{1}\Gamma_{11}^{1}+\left(\Gamma_{12}^{1}\right)_{u}\\ &+\Gamma_{12}^{2}\Gamma_{12}^{1}+A_{12}\left(\frac{FA_{21}-GA_{11}}{\det(M)}\right).\end{aligned}$$

Using the fact that $\Gamma_{ij}^{k}=\Gamma_{ji}^{k}$ for all $i,j,k=1,2$, we obtain

$$\left(\Gamma_{11}^{1}\right)_{v}-\left(\Gamma_{12}^{1}\right)_{v}=\Gamma_{12}^{2}\Gamma_{12}^{1}-\Gamma_{11}^{2}\Gamma_{22}^{1}+\frac{\left(A_{12}A_{21}-A_{11}A_{22}\right)F}{\det(M)}.$$

With $K=\frac{\det(\Sigma)}{\det(M)}$ we can write

$$KF=\Gamma_{12}^{2}\Gamma_{12}^{1}-\Gamma_{11}^{2}\Gamma_{22}^{1}+\left(\Gamma_{12}^{1}\right)_{u}-\left(\Gamma_{11}^{1}\right)_{v}=X.$$

Similarly, comparing the coefficients of ϕ_v yields

$$KE = \Gamma_{11}^1\Gamma_{21}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{12}^2\Gamma_{12}^2 + \left(\Gamma_{11}^2\right)_v - \left(\Gamma_{12}^2\right)_u = Y.$$

Therefore,

$$\begin{aligned} K &= \frac{K\det(M)}{\det(M)} \\ &= \frac{K\left(EG - F^2\right)}{\det(M)} \\ &= \frac{YG - XF}{\det(M)}, \end{aligned}$$

as required. ☐

Remark 3.14.2.

1. *It follows that* $\det(\Sigma)$ *is an intrinsic quantity, despite* Σ *itself being extrinsic.*
2. *As a plane has* $K \equiv 0$ *and a sphere of radius* r *has* $K \equiv \frac{1}{r^2}$*, it follows from Theorem 3.14.1 that the plane and sphere are not locally isometric. More generally, for regular surfaces* $S_1, S_2 \subseteq \mathbb{R}^3$ *with Gauss curvatures* K_1 *and* K_2 *respectively, if* $F : S_1 \to S_2$ *is a local isometry then* $K_2 \circ F = K_1$.

3.15 Surfaces of Constant Curvature

Theorem 3.15.1. *Let* S *be a compact connected regular oriented surface. If* S *has constant positive Gauss curvature, say* $K > 0$*, then* S *is a sphere.*

Proof. Let $\lambda_1(x) \leq \lambda_2(x)$ be the principal curvatures for $x \in S$. By the compactness of S there exists a $p \in S$ such that

$$\lambda_2(p) = \max_{x\in S} \lambda_2(x).$$

Since $\lambda_1(x)\lambda_2(x) = K > 0$, it follows that

$$\lambda_1(p) = \min_{x\in S} \lambda_1(x).$$

By a rigid motion in $\mathbb{R}^3$ and a translation we can assume without loss of generality that $p = (0,0,0)$ and the principal curvatures at p are $X_1 = (1,0,0)$ and $X_2 = (0,1,0)$. In particular, near p, from Proposition 3.10.2 we know that S is the graph of a function of the form

$$F(u,v) = \frac{\lambda_1(p)u^2 + \lambda_2(p)v^2}{2} + R(u,v).$$

Consider the chart $\phi : U \to S$ given by

$$\phi(u,v) = (u, v, F(u,v)).$$

Note that

$$\begin{cases} \phi_u = (1, 0, F_u) \\ \phi_v = (0, 1, F_v) \\ \phi_{uu} = (0, 0, F_{uu}) \\ \phi_{uv} = (0, 0, F_{uv}) \\ \phi_{uv} = (0, 0, F_{vv}). \end{cases}$$

Thus, the unit normal to S near p is

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(-F_u, -F_v, 1)}{\sqrt{1 + |\nabla F|^2}}$$

and the second fundamental form is

$$\Sigma = \begin{pmatrix} \langle N, \phi_{uu} \rangle & \langle N, \phi_{uv} \rangle \\ \langle N, \phi_{vu} \rangle & \langle N, \phi_{vv} \rangle \end{pmatrix} = \frac{1}{\sqrt{1+|\nabla F|^2}} \begin{pmatrix} F_{uu} & F_{uv} \\ F_{vu} & F_{vv} \end{pmatrix}.$$

At each $(u, v) \in U$ consider

$$e_1(u, v) = \frac{\phi_u}{|\phi_u|} = \frac{(1, 0, F_u(u, v))}{\sqrt{1 + F_u^2(u, v)}} \in T_{\phi(u,v)}S$$

and

$$e_2(u, v) = \frac{\phi_v}{|\phi_v|} = \frac{(0, 1, F_v(u, v))}{\sqrt{1 + F_v^2(u, v)}} \in T_{\phi(u,v)}S.$$

Then consider,

$$\begin{aligned} h_1(t) &:= A_{\phi(0,t)}(e_1(0, t), e_1(0, t)) \\ &= \frac{1}{1 + F_u^2(0, t)} A_{\phi(0,t)}(\phi_u(0, t), \phi_u(0, t)) \\ &= \frac{1}{1 + F_u^2(0, t)} \langle N(0, t), \phi_{uu}(0, t) \rangle \\ &= \frac{1}{1 + F_u^2(0, t)} \frac{F_{uu}(0, t)}{\sqrt{1 + |\nabla F(0, t)|^2}} \end{aligned}$$

and similarly,

$$\begin{aligned} h_2(t) &:= A_{\phi(t,0)} \left(e_2(t, 0), e_2(t, 0)\right) \\ &= \frac{1}{1 + F_v^2(0, t)} \frac{F_{vv}(0, t)}{\sqrt{1 + |\nabla F(0, t)|^2}}. \end{aligned}$$

Note that

$$h_1(0) = A_p((1, 0, 0), (1, 0, 0)) = \lambda_1(p)$$

and

$$h_2(0) = A_p((0, 1, 0), (0, 1, 0)) = \lambda_2(p).$$

Since λ_1 is minimised at p, and λ_2 is maximized at p it follows by Proposition 3.6.8 that

$$\begin{aligned} h_1(0) &= \lambda_1(p) \\ &\leq \lambda_1(\phi(0, t)) \\ &= \min_{x \in T_{\phi(0,t)},\, |x|=1} A_{\phi(0,t)}(x, x) \\ &\leq A_{\phi_{(0,t)}}(e_1(0, t), e_1(0, t)) \\ &= h_1(t). \end{aligned}$$

Similarly,

$$h_2(0) = \lambda_2(p) \geq h_2(t).$$

Thus $h_1(t)$ has a local minimum and $h_2(t)$ has a local maximum at $t = 0$. Thus, $h_1''(0) \geq 0$ and $h_2''(0) \leq 0$, meaning

$$h_1''(0) - h_2''(0) \geq 0. \tag{3.15.1}$$

Recall that

$$\nabla F(u, v) = (\lambda_1(p)u, \lambda_2(p)v) + O\left(|u|^2 + |v|^2\right)$$

so that

$$|\nabla F(u, v)|^2 = \lambda_1^2(p)u^2 + \lambda_2^2(p)v^2 + O\left(|u|^3 + |v|^3\right).$$

Therefore,

$$\begin{aligned}
h_1(t) &= \frac{1}{1+O\left(t^4\right)} \frac{F_{uu}(0,t)}{\sqrt{1+\lambda_2^2(p)t^2+O\left(t^3\right)}} \\
&= \left(1-O\left(t^4\right)\right)\left(1-\frac{1}{2}\lambda_2^2(p)t^2+O\left(t^3\right)\right)F_{uu}(0,t) \\
&= \left(1-\frac{1}{2}\lambda_2^2(p)t^2\right)F_{uu}(0,t)+O\left(t^3\right).
\end{aligned}$$

Similarly,

$$h_2(t) = \left(1-\frac{1}{2}\lambda_1^2(p)t^2\right)F_{vv}(t,0)+O\left(t^3\right).$$

Therefore,

$$\begin{aligned}
h_1'(t)-h_2'(t) = &\left(\left(1-\frac{\lambda_2^2(p)t^2}{2}\right)F_{uuv}(0,t)-\lambda_2^2(p)tF_{uu}(0,t)+O\left(t^2\right)\right) \\
&-\left(\left(1-\frac{\lambda_1^2(p)t^2}{2}\right)F_{vvu}(t,0)-\lambda_1^2(p)tF_{vv}(t,0)+O\left(t^2\right)\right)
\end{aligned}$$

and thus

$$\begin{aligned}
h_1''(t)-h_2''(t) = &\left(\left(1-\frac{\lambda_2^2(p)t^2}{2}\right)F_{uuvv}(0,t)-2\lambda_2^2(p)tF_{uuv}(0,t)-\lambda_2^2(p)F_{uu}(0,t)+O(t)\right) \\
&-\left(\left(1-\frac{\lambda_1^2(p)t^2}{2}\right)F_{vvuu}(t,0)-2\lambda_1^2(p)tF_{vvu}(t,0)-\lambda_1^2(p)F_{vv}(t,0)+O(t)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
h_1''(0)-h_2''(0) &= \lambda_1^2(p)F_{vv}(0,0)-\lambda_2^2(p)F_{uu}(0,0) \\
&= \lambda_1^2(p)\lambda_2(p)-\lambda_2^2(p)\lambda_1(p) \\
&= \lambda_1(p)\lambda_2(p)(\lambda_1(p)-\lambda_2(p)) \\
&= K(p)(\lambda_1(p)-\lambda_2(p)).
\end{aligned}$$

With (3.15.1) it follows that

$$K(p)(\lambda_1(p)-\lambda_2(p)) \geq 0.$$

This implies that $\lambda_1(p)-\lambda_2(p)\geq 0$ as $K(p)>0$ and so $\lambda_1(p)\geq\lambda_2(p)$. However, as we assumed $\lambda_1(p)\leq\lambda_2(p)$ it follows that $\lambda_1(p)=\lambda_2(p)$. Thus, we deduce that $\lambda_1(x)=\lambda_2(x)$ for all $x\in S$. Therefore, S is totally umbilic with non-zero curvature and so from Theorem 3.9.3 it must be contained within a sphere. Since S is compact it must be the case that S is a sphere. □

3.16 Integration of Surfaces

Let $S\subseteq\mathbb{R}^3$ be a regular surface, and let $\phi: U\to S$ be a chart. For simplicity let $0\in U$ and assume that $0<\delta_u,\delta_v\ll 1$ are such that $R:=[0,\delta_u]\times[0,\delta_v]\subseteq U$. In particular, note that R has area $\delta_u\delta_v$. Under the transformation ϕ the rectangle R is approximately mapped to the parallelogram P spanned by $\phi_u(0,0)$ and $\phi_v(0,0)$ scaled by δ_u and δ_v respectively. More specifically, P is the parallelogram with corners

$$\{p, p+\delta_u\phi_u(0,0), p+\delta_v\phi_v(0,0), p+\delta_u\phi_u(0,0)+\delta_v\phi_v(0,0)\}$$

where $p:=\phi(0,0)$. In particular, P has area given by

$$\mathrm{area}(P) = \left|\delta_u\phi_u(0,0)\times\delta_v\phi_v(0,0)\right| = \delta_u\delta_v\left|\phi_u(0,0)\times\phi_v(0,0)\right|.$$

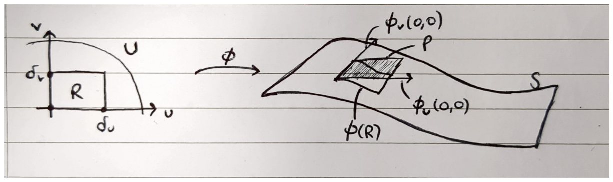


Figure 3.16.1: Area element on a surface.

Definition 3.16.1. *Given $D \subseteq U$ compact, let*

$$\operatorname{area}(\phi(D)) = \int_D |\phi_u(u,v) \times \phi_v(u,v)| \,\mathrm{d}u\,\mathrm{d}v.$$

Proposition 3.16.2. *Let S be a regular surface with charts $\phi : U \to S$ and $\psi : U' \to S$. Suppose that for compact sets $D \subseteq U$ and $D' \subseteq U'$ we have $\phi(D) = \psi(D')$. Then,*

$$\operatorname{area}(\phi(D)) = \operatorname{area}(\psi(D')).$$

Proof. Consider $f : U' \to U$ given by $f = \phi^{-1} \circ \psi$, and in particular assume that

$$f(u,v) = (x(u,v), y(u,v)).$$

Then

$$\psi(u,v) = (\phi \circ f)(u,v) = \phi(x(u,v), y(u,v))$$

and so

$$\begin{aligned}
\frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v} &= \left(\frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u}\right) \times \left(\frac{\partial \phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial v}\right) \\
&= \left(\frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y}\right)\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)
\end{aligned}$$

where we have used the bilinearity and anti-symmetry properties of the cross-product. Therefore, recalling the change of coordinates formula, it follows that

$$\begin{aligned}
\operatorname{area}(\psi(D')) &= \int_{D'} \left|\frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v}\right| \mathrm{d}u\,\mathrm{d}v \\
&= \int_{D'} \left|\frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y}\right| \left|\det\left(\begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix}\right)\right| \mathrm{d}u\,\mathrm{d}v \\
&= \int_{D} \left|\frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y}\right| \mathrm{d}x\,\mathrm{d}y \\
&= \operatorname{area}(\phi(D)).
\end{aligned}$$

□

The area of a surface can also be expressed in terms of the first fundamental form.

Proposition 3.16.3. *Let $\phi : U \to S$ be a chart for a regular surface S. Then for a compact set $D \subseteq U$ we have*

$$\begin{aligned}
\operatorname{area}(\phi(D)) &= \int_D \sqrt{\det(M(u,v))}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_D \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}$$

where

$$M = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} |\phi_u|^2 & \langle \phi_u, \phi_v \rangle \\ \langle \phi_u, \phi_v \rangle & |\phi_v|^2 \end{pmatrix}.$$

Proof. Let $v, w \in \mathbb{R}^3$ with v non-zero. Then, v is orthogonal to $w - \frac{\langle v,w \rangle}{|v|^2} v$ and so

$$\begin{aligned} |v \times w|^2 &= \left| v \times \left(w - \frac{\langle v, w \rangle v}{|v|^2} \right) \right|^2 \\ &= |v|^2 \left| w - \frac{\langle v, w \rangle v}{|v|^2} \right|^2 \\ &= |v|^2 \left(|w|^2 - 2 \frac{\langle v, w \rangle \langle v, w \rangle}{|v|^2} + \left(\frac{\langle v, w \rangle}{|v|^2} \right)^2 |v|^2 \right) \\ &= |v|^2 |w|^2 - \langle v, w \rangle^2. \end{aligned}$$

In particular, with $v = \phi_u$ and $w = \phi_v$ it follows that

$$|\phi_u \times \phi_v|^2 = |\phi_u|^2 \, |\phi_v|^2 - \langle \phi_u, \phi_v \rangle^2 = EG - F^2.$$

Therefore,

$$\begin{aligned} \operatorname{area}(\phi(D)) &= \int_D |\phi_u \times \phi_v| \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_D \sqrt{EG - F^2} \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_D \sqrt{\det(M(u, v))} \, \mathrm{d}u \, \mathrm{d}v. \end{aligned}$$

□

Definition 3.16.4. *Let S be a regular surface, let $f : S \to \mathbb{R}$ be smooth, $\phi : U \to S$ a chart and $D \subseteq U$ compact. Then the integral of f on $\phi(D)$ is given by*

$$\int_{\phi(D)} f \, \mathrm{d}A := \int_D (f \circ \phi)(u, v) \sqrt{EG - F^2} \, \mathrm{d}u \, \mathrm{d}v.$$

If S is a compact surface, then we can write $S = \bigcup_{i=1}^k S_i$ such that the following hold.

- For $i \neq j$ we have $S_i \cap S_j \subseteq \partial S_i \cap \partial S_j$.
- For $i = 1, \ldots, k$ there is a chart $\phi_i : U_i \to S$ and $D_i \subseteq U_i$ such that $\phi_i(D_i) = S_i$.

Hence, we let

$$\begin{aligned} \int_S f \, \mathrm{d}A &:= \sum_{i=1}^k \int_{\phi(D_i)} f \, \mathrm{d}A \\ &= \sum_{i=1}^k \int_{D_i} (f \circ \phi_i)(u, v) \sqrt{E_i G_i - F_i^2} \, \mathrm{d}u \, \mathrm{d}v. \end{aligned}$$

This is independent of the chosen decomposition of S.

Example 3.16.5. *Let $\psi : (\epsilon, 2\pi - \epsilon) \times (\epsilon, \pi - \epsilon) \to \mathrm{S}^2$ be given by*

$$\psi(\theta, \phi) = (\cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi)).$$

Note that ψ is a chart for the surface

$$\tilde{\mathrm{S}}^2 := \psi((\epsilon, 2\pi - \epsilon) \times (\epsilon, \pi - \epsilon)).$$

We cannot construct a chart for S^2 directly since the poles and meridian lines cause problems, the surface $\tilde{\mathrm{S}}^2$ avoids these points. Observe that

$$\begin{cases} \frac{\partial \psi}{\partial \theta} = (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0) \\ \frac{\partial \psi}{\partial \phi} = (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi)) \end{cases}$$

such that

$$\frac{\partial \psi}{\partial \theta} \times \frac{\partial \psi}{\partial \phi} = \left(-\cos(\theta)\sin^2(\phi), -\sin(\theta)\sin^2(\phi), -\cos(\phi)\sin(\phi)\right).$$

In particular,

$$\begin{aligned} \left|\frac{\partial \psi}{\partial \theta} \times \frac{\partial \psi}{\partial \phi}\right| &= \sqrt{\cos^2(\theta)\sin^4(\phi) + \sin^2(\theta)\sin^4(\phi) + \cos^2(\phi)\sin^2(\phi)} \\ &= \sqrt{\sin^2(\phi)\left(\sin^2(\phi) + \cos^2(\phi)\right)} \\ &= \sin(\phi). \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{area}\left(\tilde{\mathrm{S}}^2\right) &= \int_{\epsilon}^{2\pi-\epsilon} \int_{\epsilon}^{\pi-\epsilon} \sin(\phi)\, \mathrm{d}\phi\, \mathrm{d}\theta \\ &= (2\pi - 2\epsilon)(2\cos(\epsilon)). \end{aligned}$$

Strictly speaking, we should only compute the area using compact sets, which $(\epsilon, 2\pi - \epsilon) \times (\epsilon, \pi - \epsilon)$ is not. However, this detail turns out to be insignificant, so in any case sending $\epsilon \to 0$ it follows that

$$\operatorname{area}\left(\mathrm{S}^2\right) = 4\pi$$

as expected.

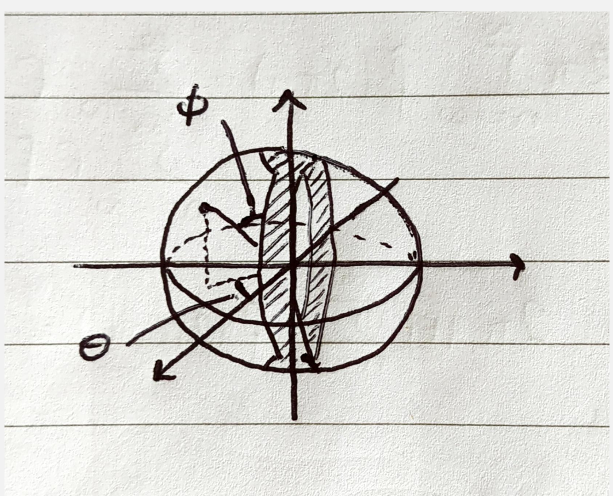


Figure 3.16.2

Exercise 3.16.6. *Let $S \subseteq \mathbb{R}^3$ be a compact, connected, nonempty surface with curvature K. Suppose $K > 0$ everywhere on S. Show that $\int_S K \, \mathrm{d}A \geq 4\pi$.*

3.17 Solution to Exercises

Exercise 3.4.2

Solution. Suppose that $\psi : U' \to S$ is another chart for $p \in S$. More specifically, $\psi(U') = V \cap S$, where $V \subseteq \mathbb{R}^3$ is an open neighbourhood of p with $\phi(U) = V \cap S$. Then as $\phi : U \to V \cap S$ is a homeomorphism we can write

$$F \circ \psi = F \circ \left(\phi \circ \phi^{-1}\right) \circ \psi : U' \to S.$$

As $\mathrm{d}\phi_q$ is invertible it follows that $\phi^{-1} \circ \psi$ is smooth. Therefore, as $F \circ \phi$ is smooth it follows that $F \circ \psi$ is smooth. In particular, this means that $F \circ \psi$ is smooth for any chart ψ, and so $F : S \to \mathbb{R}^n$ is smooth as per statement 1 of Definition 3.4.1. $\square$

Exercise 3.5.7

Proof. Observe that

$$\frac{\partial \phi}{\partial u} = \left(1, 0, -\frac{u}{\sqrt{1-u^2-v^2}}\right)$$

and

$$\frac{\partial \phi}{\partial v} = \left(0, 1, -\frac{v}{\sqrt{1-u^2-v^2}}\right).$$

So that,

$$\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} = \left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1\right).$$

Noting that

$$\left|\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}\right| = \frac{1}{\sqrt{1-u^2-v^2}}$$

it follows that

$$\begin{aligned}
N(\phi(u,v)) &= \frac{\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}}{\left|\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}\right|} \\
&= \sqrt{1-u^2-v^2}\left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1\right) \\
&= \left(u, v, \sqrt{1-u^2-v^2}\right).
\end{aligned}$$

Therefore, the Gauss map S^2 is the identity map. $\square$

Exercise 3.6.10

Solution. Let $\phi : U \to S$ be a chart for S at p such that $U \subseteq \mathbb{R}^2$ is connected. Then for any $p' \in \phi(U)$ there exists a smooth path $\gamma : [0,1] \to U$ with $\gamma(0) = \phi^{-1}(p)$ and $\gamma(1) = \phi^{-1}(q')$. In particular, $\phi \circ \gamma : [0,1] \to S$ is a smooth path from p to p'. As $\lambda_1 = \lambda_2 \equiv 0$ we have that $\mathrm{d}N_p \equiv 0$, thus $N \equiv v$ for some $v \in \mathrm{S}^2$. Therefore, as $\frac{\mathrm{d}}{\mathrm{d}t}(\phi \circ \gamma)(t) \in T_{\phi(\gamma(t))}S$ it must be perpendicular to v. Hence,

$$0 = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\phi(\gamma(t))), v \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle \phi(\gamma(t)), v \rangle.$$

Hence, $\langle \phi(\gamma(t)), v \rangle$ is constant and so $\langle p', v \rangle = \langle p, v \rangle$. As S is connected it follows that $\langle p', v \rangle = \langle p, v \rangle$ for all $p' \in S$, which means that S lies in a plane. $\square$

Exercise 3.8.4

Solution.

1. A chart for the helicoid S is
$$\phi(u,v) = (u\cos(v), u\sin(v), v).$$
In particular,
$$\begin{cases} x = u\cos(v) \\ y = u\sin(v) \\ z = v. \end{cases}$$
Observe that
$$\frac{\partial\phi}{\partial u} = (\cos(v), \sin(v), 0)$$
and
$$\frac{\partial\phi}{\partial v} = (-u\sin(v), u\cos(v), 1).$$
Hence,
$$\begin{aligned} N(u,v) &= \frac{\frac{\partial\phi}{\partial u}(u,v) \times \frac{\partial\phi}{\partial v}(u,v)}{\left|\frac{\partial\phi}{\partial u}(u,v) \times \frac{\partial\phi}{\partial v}(u,v)\right|} \\ &= \frac{1}{\sqrt{u^2+1}}(\sin(v), -\cos(v), u). \end{aligned}$$
In particular,
$$N(x,y,z) = \frac{1}{\sqrt{x^2+y^2+1}}\left(\sin(z), -\cos(z), \sqrt{x^2+y^2}\right).$$
For $(v_1, v_2, v_3) \in T_{(x,y,z)}S$, let $\gamma : (-\epsilon, \epsilon) \to S$ be a smooth map such that $\gamma(0) = (x,y,z)$ and $\gamma'(0) = (v_1, v_2, v_3)$. Note that
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sin(\gamma_3(t))}{\sqrt{\gamma_1(t)^2+\gamma_2(t)^2+1}}\right)\Bigg|_{t=0} = \frac{\left(x^2+y^2+1\right)\cos(z)v_3 - \sin(z)(xv_1+yv_2)}{(x^2+y^2+1)^{\frac{3}{2}}}$$
and similarly,
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\cos(\gamma_3(t))}{\sqrt{\gamma_1(t)^2+\gamma_2(t)^2+1}}\right)\Bigg|_{t=0} = \frac{\left(x^2+y^2+1\right)\sin(z)v_3 + \cos(z)(xv_1+yv_2)}{(x^2+y^2+1)^{\frac{3}{2}}}.$$
Moreover,
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sqrt{\gamma_1(t)^2+\gamma_2(t)^2}}{\sqrt{\gamma_1(t)^2+\gamma_2(t)^2+1}}\right)\Bigg|_{t=0} = \frac{xv_1+yv_2}{(x^2+y^2+1)^{\frac{3}{2}}\sqrt{x^2+y^2}},$$
therefore,
$$\begin{aligned} \mathrm{d}N_{(x,y,z)}(v_1,v_2,v_3) = \Bigg(& \frac{\left(x^2+y^2+1\right)\cos(z)v_3 - \sin(z)(xv_1+yv_2)}{(x^2+y^2+1)^{\frac{3}{2}}}, \\ & \frac{\left(x^2+y^2+1\right)\sin(z)v_3 + \cos(z)(xv_1+yv_2)}{(x^2+y^2+1)^{\frac{3}{2}}}, \\ & \frac{xv_1+yv_2}{(x^2+y^2+1)^{\frac{3}{2}}\sqrt{x^2+y^2}}\Bigg). \end{aligned}$$
Let $Y_1 = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0\right)$ and $Y_2 = (-y, x, 1)$. Then
$$\langle Y_1, N(x,y,z)\rangle = \langle Y_2, N(x,y,z)\rangle = 0.$$

Moreover,

$$\mathrm{d}N_{(x,y,z)}(Y_1) = \frac{1}{(x^2+y^2+1)^{\frac{3}{2}}} Y_2$$

and

$$\mathrm{d}N_{(x,y,z)}(Y_2) = \frac{1}{\sqrt{x^2+y^2+1}} Y_1.$$

Therefore, $X_1 := Y_1 - \sqrt{x^2+y^2+1} Y_2$ and $X_2 := \sqrt{x^2+y^2+1} Y_1 + Y_2$ are tangent vectors to the helicoid at (x,y,z). Moreover,

$$\mathrm{d}N_{(x,y,z)}(X_1) = -\frac{1}{x^2+y^2+1} X_1$$

and

$$\mathrm{d}N_{(x,y,z)}(X_2) = \frac{1}{x^2+y^2+1} X_2.$$

Hence, X_1 and X_2 are the principal directions with $\lambda_1(x,y,z) = -\frac{1}{x^2+y^2+1}$ and $\lambda_2(x,y,z) = \frac{1}{x^2+y^2+1}$. Therefore, $K(x,y,z) = \frac{1}{(x^2+y^2+1)^2}$ and $H(x,y,z) = 0$. This means that the helicoid is a minimal surface.

2. A chart for the catenoid S is

$$\phi(u,v) = (\cosh(u)\cos(v), \cosh(u)\sin(v), u).$$

In particular,

$$\begin{cases} x = \cosh(u)\cos(v) \\ y = \cosh(u)\sin(v) \\ z = u. \end{cases}$$

Observe that

$$\frac{\partial \phi}{\partial u} = (\sinh(u)\cos(v), \sinh(u)\sin(v), 1)$$

and

$$\frac{\partial \phi}{\partial v} = (-\cosh(u)\sin(v), \cosh(u)\cos(v), 0).$$

Hence,

$$\begin{aligned} N(u,v) &= \frac{\frac{\partial \phi}{\partial u}(u,v) \times \frac{\partial \phi}{\partial v}(u,v)}{\left|\frac{\partial \phi}{\partial u}(u,v) \times \frac{\partial \phi}{\partial v}(u,v)\right|} \\ &= \frac{1}{\cosh^2(u)} \left(-\cosh(u)\cos(v), -\cosh(u)\sin(v), \sinh(u)\cosh(u)\right). \end{aligned}$$

In particular,

$$N(x,y,z) = \left(-\frac{x}{\cosh^2(z)}, -\frac{y}{\cosh^2(z)}, \tanh(z)\right).$$

For $(v_1,v_2,v_3) \in T_{(x,y,z)}S$, let $\gamma : (-\epsilon,\epsilon) \to S$ be a smooth map such that $\gamma(0) = (x,y,z)$ and $\gamma'(0) = (v_1,v_2,v_3)$. Note that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\gamma_1(t)}{\cosh^2(\gamma_3(t))}\right)\Big|_{t=0} &= \left(\frac{\cosh^2(\gamma_3(t))\gamma_1'(t) - 2\cosh^2(\gamma_3(t))\gamma_3'(t)\gamma_1(t)}{\cosh^4(\gamma_3(t))}\right) \\ &= \frac{2v_3 x - v_1}{\cosh^2(z)}. \end{aligned}$$

and similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\gamma_2(t)}{\cosh^2(\gamma_3(t))}\right)\Big|_{t=0} = \frac{2v_3 y - v_2}{\cosh^2(z)}.$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tanh(\gamma_3(t))\right)\Big|_{t=0} = \left(\frac{\gamma_3'(t)}{\cosh^2(\gamma_3(t))}\right)\Big|_{t=0} = \frac{v_3}{\cosh^2(z)},$$

therefore,

$$\mathrm{d}N_{(x,y,z)}(v_1, v_2, v_3) = \left(\frac{2v_3x - v_1}{\cosh^2(z)}, \frac{2v_3y - v_2}{\cosh^2(z)}, \frac{v_3}{\cosh^2(z)}\right).$$

Note that

$$\langle (x, -y, 0), N(x, y, z)\rangle = 0$$

and

$$\mathrm{d}N_{(x,y,z)}(x, -y, 0) = -\frac{1}{\cosh^2(z)}(x, -y, 0).$$

Similarly, note that

$$\left\langle \left(\frac{1}{2}\sinh(z)\cosh(z), \frac{1}{2}\sinh(z)\cosh(z), 1\right), N(x, y, z)\right\rangle = 0$$

and

$$\mathrm{d}N_{(x,y,z)}\left(\frac{1}{2}\sinh(z)\cosh(z), \frac{1}{2}\sinh(z)\cosh(z), 1\right) = \frac{1}{\cosh^2}\left(\frac{1}{2}\sinh(z)\cosh(z), \frac{1}{2}\sinh(z)\cosh(z), 1\right).$$

Therefore, $\lambda_1(x, y, z) = -\frac{1}{\cosh^2(z)}$ and $\lambda_2(x, y, z) = \frac{1}{\cosh^2(z)}$. Hence, $K(x, y, z) - \frac{1}{\cosh^4(z)}$ and $H(x, y, z) = 0$. This means that the catenoid is a minimal surface.

□

Exercise 3.8.7

Solution.

1. Let $p = \gamma(t)$ for $t \in I$. Then $\gamma'(t) \in T_pS$ and $|\gamma'(t)| = 1$. Using Theorem 3.7.1 it follows that $A_p(\gamma'(t), \gamma'(t)) = 0$. Therefore, using the characterisations of the principal curvatures provided by Proposition 3.6.8 it follows that $\lambda_1(p) \le 0$ and $\lambda_2(p) \ge 0$. Therefore, $K(p) = \lambda_1(p)\lambda_2(p) \le 0$.

2. Let $\{T_\gamma, N_\gamma, B_\gamma\}$ be the orthonormal Frenet frame at $p = \gamma(t_0)$ for $t_0 \in I$. Since γ is parameterised by arc length we have $T_\gamma = \gamma'(t_0)$. Thus, $N_\gamma = \frac{1}{k(t_0)}\gamma''(t_0) = \frac{1}{k(t_0)}\mathbf{k}(t_0)$. Therefore,

$$\langle N_\gamma, N(\gamma(t_0))\rangle = \frac{1}{k(t_0)}\langle \mathbf{k}(t_0), N(\gamma(t_0))\rangle = 0.$$

Meaning N_γ is orthogonal to $N(\gamma(t_0))$ and so $\{T_\gamma, N_\gamma\}$ forms an orthonormal basis for T_pS. Furthermore, as $N(\gamma(t_0))$ is also orthogonal to T_γ it follows that $N(\gamma(t_0)) = \pm B_\gamma$. Note that

$$\begin{aligned}\langle \mathrm{d}N_p(T_\gamma), T_\gamma\rangle &= \left\langle \frac{\mathrm{d}}{\mathrm{d}t}N(\gamma(t))\Big|_{t=t_0}, T_\gamma\right\rangle \\ &= \left\langle \pm B_\gamma', T_\gamma\right\rangle \\ &= \mp\tau(t_0)\left\langle N_\gamma, T_\gamma\right\rangle \\ &= 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle \mathrm{d}N_p(T_\gamma), N_\gamma\rangle &= \left\langle \pm B_\gamma', N_\gamma\right\rangle \\ &= \mp\tau(t_0)\left\langle N_\gamma, N_\gamma\right\rangle \\ &= \mp\tau(t_0).\end{aligned}$$

Therefore, the matrix of $\mathrm{d}N_p$ with respect to the basis $\{T_\gamma, N_\gamma\}$ is

$$S_p = \begin{pmatrix} 0 & \mp\tau(t_0) \\ \mp\tau(t_0) & \langle \mathrm{d}N_p(N_\gamma), N_\gamma\rangle \end{pmatrix}.$$

Therefore,

$$K(p) = \det(S_p) = -|\tau(t_0)|$$

and so

$$|\tau(t_0)| = \sqrt{-K(p)}.$$

□

Exercise 3.10.1

Solution. Observe that

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \gamma'(t), N(\gamma(t))\right\rangle \\ &= \left\langle \gamma''(t), N(\gamma(t))\right\rangle + \left\langle \gamma'(t), \mathrm{d}N_{\gamma(t)}\left(\gamma'(t)\right)\right\rangle \\ &= \left\langle \gamma''(t), N(\gamma(t))\right\rangle - A_{\gamma(t)}\left(\gamma'(t), \gamma'(t)\right). \end{aligned}$$

□

Exercise 3.11.6

Solution.

1. With the chart of Exercise 3.8.4 we have that

   $$\frac{\partial \phi}{\partial u} = (\cos(v), \sin(v), 0)$$

   and

   $$\frac{\partial \phi}{\partial v} = (-u\sin(v), u\cos(v), 1)$$

   so that,

   $$N(\phi(u,v)) = \frac{1}{\sqrt{u^2+1}}(\sin(v), -\cos(v), u).$$

   Moreover,

   $$\begin{cases} \frac{\partial^2\phi}{\partial u^2} = (0,0,0) \\ \frac{\partial^2\phi}{\partial u \partial v} = (-\sin(v), \cos(v), 0) \\ \frac{\partial^2\phi}{\partial v^2} = (-u\cos(v), -u\sin(v), 0) \end{cases}$$

   so that,

   $$M = \begin{pmatrix} 1 & 0 \\ 0 & u^2+1 \end{pmatrix}$$

   and

   $$\Sigma = \begin{pmatrix} 0 & -\frac{1}{\sqrt{u^2+1}} \\ -\frac{1}{\sqrt{u^2+1}} & 0 \end{pmatrix}.$$

   Therefore,

   $$K(p) = \left(-\frac{1}{u^2+1}\right)\frac{1}{u^2+1} = -\frac{1}{\left(u^2+1\right)^2}$$

   and

   $$H(p) = \frac{1}{2}\mathrm{tr}\left(\frac{1}{u^2+1}\begin{pmatrix} u^2+1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -\frac{1}{\sqrt{u^2+1}} \\ -\frac{1}{\sqrt{u^2+1}} & 0 \end{pmatrix}\right) = 0.$$

2. With the chart of Exercise 3.8.4 we have that
$$\frac{\partial \phi}{\partial u} = (\sinh(u)\cos(v), \sinh(u)\sin(v), 1)$$
and
$$\frac{\partial \phi}{\partial v} = (-\cosh(u)\sin(v), \cosh(u)\cos(v), 0)$$
so that,
$$N(\phi(u,v)) = \frac{1}{\cosh^2(u)}\left(-\cosh(u)\cos(v), -\cosh(u)\sin(v), \sinh(u)\cosh(u)\right).$$
Moreover,
$$\begin{cases} \frac{\partial^2\phi}{\partial u^2} = (\cosh(u)\cos(v), \cosh(u)\sin(v), 0) \\ \frac{\partial^2\phi}{\partial u\partial v} = (-\sinh(u)\sin(v), \sinh(u)\cos(v), 0) \\ \frac{\partial^2\phi}{\partial v^2} = (-\cosh(u)\cos(v), -\cosh(u)\sin(v), 0) \end{cases}$$
so that,
$$M = \begin{pmatrix} \cosh^2(u) & 0 \\ 0 & \cosh^2(u) \end{pmatrix}$$
and
$$\Sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$
Therefore,
$$K(p) = -\frac{1}{\cosh^4(u)}$$
and
$$H(p) = \frac{1}{2}\mathrm{tr}\left(\frac{1}{\cosh^4(u)}\begin{pmatrix} \cosh^2 & 0 \\ 0 & \cosh^2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0.$$
$\square$

Exercise 3.12.7

Solution. We have the charts $\varphi_1 : \mathbb{R} \times (0, 2\pi) \to S_1$ given by
$$\varphi_1(u,v) = (\cosh(u)\cos(u), \cosh(u)\sin(v), u)$$
and $\varphi_2 : \mathbb{R} \times (0, 2\pi) \to S_2$ given by
$$\varphi_2(u,v) = (u\cos(v), u\sin(v), v).$$
Let $\varphi(u,v) := \varphi_1(u,v) : U \to S_1$ and $\tilde{\varphi}(u,v) := \varphi_2(\sinh(u), v) : U \to S_2$, where $U := \mathbb{R} \times (0, 2\pi)$. Observe that
$$\begin{cases} \varphi_u = (\sinh(u)\cos(v), \sinh(u)\sin(v), 1) \\ \varphi_v = (-\cosh(u)\sin(v), \cosh(u)\cos(v), 0) \end{cases}$$
and
$$\begin{cases} \tilde{\varphi}_u = (\cosh(u)\cos(v), \cosh(u)\sin(v), 0) \\ \tilde{\varphi}_v = (-\cosh(u)\sin(v), \cosh(u)\cos(v), 0). \end{cases}$$
Therefore,
$$\begin{aligned} M &= \begin{pmatrix} \sinh^2(u) + 1 & 0 \\ 0 & \cosh^2(u) \end{pmatrix} \\ &= \begin{pmatrix} \cosh^2(u) & 0 \\ 0 & \sinh^2(u) \end{pmatrix} \\ &= \tilde{M}. \end{aligned}$$
Therefore, using Lemma 3.12.6 it follows that S_1 and S_2 are locally isometric. $\square$

Exercise 3.13.6

Solution. Note that

$$\begin{cases} \phi_u = (1, 0, f_u(u, v)) \\ \phi_v = (0, 1, f_v(u, v)) \end{cases}$$

so that

$$(g_{ij}) = \begin{pmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{pmatrix}$$

and in particular,

$$(g^{ij}) = \frac{1}{1 + f_u^2 + f_v^2} \begin{pmatrix} 1 + f_v^2 & -f_u f_v \\ -f_u f_v & 1 + f_u^2 \end{pmatrix}.$$

Therefore,

$$\begin{cases} \Gamma_{11}^1 = \frac{g^{11}}{2}(\partial_u g_{11}) + \frac{g^{12}}{2}(2\partial_u g_{12} - \partial_v g_{11}) = \frac{f_u f_{uu}}{1+f_u^2+f_v^2} \\ \Gamma_{11}^2 = \frac{g^{21}}{2}(\partial_u g_{11}) + \frac{g^{22}}{2}(2\partial_u g_{12} - \partial_v g_{11}) = \frac{f_v f_{uu}}{1+f_u^2+f_v^2} \\ \Gamma_{21}^1 = \Gamma_{12}^1 = \frac{g^{11}}{2}(\partial_v g_{11}) + \frac{g^{12}}{2}(\partial_u g_{22}) = \frac{f_u f_{uv}}{(1+f_u^2+f_v^2)} \\ \Gamma_{21}^2 = \Gamma_{12}^2 = \frac{g^{21}}{2}(\partial_v g_{11}) + \frac{g^{22}}{2}(\partial_u g_{22}) = \frac{f_v f_{uv}}{(1+f_u^2+f_v^2)} \\ \Gamma_{22}^1 = \frac{g^{11}}{2}(2\partial_v g_{21} - \partial_u g_{22}) + \frac{g^{21}}{2}(\partial_v g_{22}) = \frac{f_u f_{vv}}{1+f_u^2+f_v^2} \\ \Gamma_{22}^2 = \frac{g^{12}}{2}(2\partial_v g_{21} - \partial_u g_{22}) + \frac{g^{22}}{2}(\partial_v g_{22}) = \frac{f_v f_{vv}}{1+f_u^2+f_v^2}. \end{cases}$$

□

Exercise 3.16.6

Solution. From Proposition 3.5.10, the Gauss map $N : S \to \mathrm{S}^2$ is surjective. In particular, as $\det(\mathrm{d}N_p) = K(p) > 0$, it follows that N is a local diffeomorphism. Let $\phi : U \to S$ be a chart for $p \in S$. Then restricting U if necessary, we can assume that N is a diffeomorphism on $\phi(U)$. Therefore, $\psi = N \circ \phi : U \to \mathrm{S}^2$ is a chart for S^2 at $N(p)$. For $C \subseteq U$ compact observe that

$$\begin{aligned} \int_{\phi(C)} K \,\mathrm{d}A &= \int_C \det\left(\mathrm{d}N_{\phi(u,v)}\right) |\phi_u \times \phi_v| \,\mathrm{d}u \,\mathrm{d}v \\ &= \int_C |\psi_u \times \psi_v| \,\mathrm{d}u \,\mathrm{d}v \\ &= \operatorname{area}(\psi(C)). \end{aligned}$$

As S is compact we can write $S = \bigcup_{i=1}^k S_i$ where the S_i only intersect along their boundary, and for each S_i there exists a chart $\phi_i : U_i \to S$ such that $\phi_i(C_i) = S_i$ for a compact set $C_i \subseteq U_i$. Since $(\psi(C_i))_{i=1}^k$ covers S^2 it follows that

$$\int_S K \,\mathrm{d}A = \sum_{i=1}^k \int_{\phi(C_i)} K \,\mathrm{d}A = \sum_{i=1}^k \operatorname{area}(\psi(C_i)) \geq 4\pi.$$

□

4 Geodesics

4.1 Geodesic Curvature

A geodesic for a surface S is a regular curve $\gamma : (a,b) \to S$ which is in some sense the straightest line on S. For instance, on $\mathbb{R}^n$ the regular curve $\gamma : (a,b) \to \mathbb{R}^n$ given by $\gamma(t) = ct + d$ for $c, d \in \mathbb{R}$ is a straight line. In particular, γ can be characterised by either of the following.

1. γ minimises the distance between points on $\mathbb{R}^n$.
2. γ has zero acceleration or curvature. That is, $\gamma''(t) = 0$ for all $t \in (a,b)$.

To extrapolate these ideas beyond $\mathbb{R}^n$, we utilise the second characteristic property of straight lines in $\mathbb{R}^n$. More specifically, let S be an oriented surface with Gauss map $N : S \to \mathrm{S}^2$, and let $\gamma : I \to S \subseteq \mathbb{R}^3$, where $I \subseteq \mathbb{R}^3$ is an interval, be a regular curve parameterised by arc length. Note that $\{\gamma', N \times \gamma', N\}$ is an orthonormal basis for $\mathbb{R}^3$ with $\{\gamma'(t), N(\gamma(t)) \times \gamma'(t)\}$ an orthonormal basis for $T_{\gamma(t)}S$ for all $t \in I$. Recall that the curvature vector for γ is $\mathbf{k}(t) = \gamma''(t)$, which is orthogonal to $\gamma'(t)$, Proposition 2.3.6. Thus,

$$\mathbf{k}(t) = \langle \mathbf{k}(t), N(\gamma(t)) \times \gamma'(t) \rangle \left(N(\gamma(t)) \times \gamma'(t) \right) + \langle \mathbf{k}(t), N(\gamma(t)) \rangle N(\gamma(t)).$$

Definition 4.1.1. *With notation as above, $k_g(t) := \langle \mathbf{k}(t), N(\gamma(t)) \times \gamma'(t) \rangle$ is referred to as the geodesic curvature, and $k_n(t) := \langle \mathbf{k}(t), N(\gamma(t)) \rangle$ is referred to as the normal curvature of γ at t.*

Remark 4.1.2.

1. *Note that*
$$k(t) = |\mathbf{k}(t)| = \sqrt{k_g(t)^2 + k_n(t)^2}.$$
2. *The geodesic curvature describes the curvature of γ tangential to S.*

Definition 4.1.3. *A curve $\gamma : I \to S \subseteq \mathbb{R}^3$ is a geodesic if $k_g(t) = 0$ for all $t \in I$.*

Example 4.1.4.

1. *Let S be the x-y plane, such that $N = (0,0,1)$. Let $\gamma(t) = (x(t), y(t), 0)$ be parameterised by arc length. Then,*
$$\begin{aligned} k_n(t) &= \langle \gamma''(t), N(\gamma(t)) \rangle \\ &= \langle (x''(t), y''(t), 0), (0,0,1) \rangle \\ &= 0. \end{aligned}$$
Hence,
$$\mathbf{k}(t) = k_g \left(N(\gamma(t)) \times \gamma'(t) \right)$$
so that
$$(x''(t), y''(t), 0) = k_g \left(-y', x', 0 \right).$$
Therefore, $k_g \equiv 0$ if and only if $x''(t) \equiv y''(t) \equiv 0$. In other words, the geodesics of S are straight lines.
2. *Let S^2 be the unit sphere in $\mathbb{R}^3$, and for $r > 0$ let $\gamma : [0, 2\pi r] \to \mathrm{S}^2$ be given by*
$$\gamma(t) = \left(r \cos\left(\frac{t}{r}\right), r \sin\left(\frac{t}{r}\right), \sqrt{1 - r^2} \right),$$

which is the circle of radius r on S^2 parallel to the x-y plane. Note that γ is parameterised by arc length and recall that $N(p) = p$ for every $p \in \mathrm{S}^2$, hence,

$$N(\gamma(t)) \times \gamma'(t) = \left(r\cos\left(\frac{t}{r}\right), r\sin\left(\frac{t}{r}\right), \sqrt{1-r^2}\right) \times \left(-\sin\left(\frac{t}{r}\right), \cos\left(\frac{t}{r}\right), 0\right)$$
$$= \left(-\sqrt{1-r^2}\cos\left(\frac{t}{r}\right), -\sqrt{1-r^2}\sin\left(\frac{t}{r}\right), r\right).$$

Moreover,

$$\gamma''(t) = \left(-\frac{1}{r}\cos\left(\frac{t}{r}\right), -\frac{1}{r}\sin\left(\frac{t}{r}\right), 0\right),$$

so that

$$k_g(t) = \langle \gamma''(t), N(\gamma(t)) \times \gamma'(t)\rangle = \frac{\sqrt{1-r^2}}{r}.$$

Therefore, γ is a geodesic when $r = 1$, that is γ is the equator of the sphere. It follows that great circles of S^2 are geodesics. A great circle is a path on S^2 obtained by intersecting S^2 with a plane passing through the origin. Using Lemma 2.4.8 we can rotate the curve such that it lies on the equator of the sphere whilst preserving its curvature. Therefore, with the above computations, we deduce that a great circle is a geodesic and geodesics are great circles.

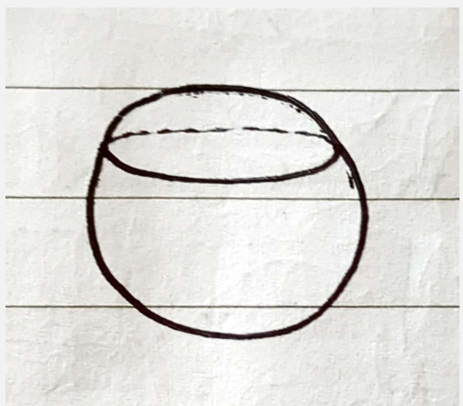


Figure 4.1.1

Let S be a regular oriented surface with chart $\phi : U \to S$. Let $\gamma : I \to S \subseteq \mathbb{R}^3$ be a regular curve parameterised by arc length, with $\gamma(I) \subseteq \phi(U)$. Then one can write $\gamma(t) = \phi\left(x^1(t), x^2(t)\right)$, where $\left(x^1(t), x^2(t)\right)$ is a curve in U. Recall, that

$$\frac{\partial^2\phi}{\partial x^i \partial x^j} = \sum_{k=1}^{2} \Gamma_{ij}^k \frac{\partial \phi}{\partial x^k} + A_{ij}N,$$

where N is the Gauss map of the surface.

Proposition 4.1.5. *For S, ϕ and γ as above, γ is a geodesic if and only if*

$$\ddot{x}^k(t) + \sum_{i,j=1}^{2} \Gamma_{ij}^k\left(x^1(t), x^2(t)\right)\dot{x}^i(t)\dot{x}^j(t) = 0 \tag{4.1.1}$$

for $k = 1, 2$.

Proof. A curve γ is a geodesic if and only if $\gamma''(t)$ is proportional to $N(\gamma(t))$. Note that

$$\gamma'(t) = \sum_{i=1}^{2} \dot{x}^i(t) \frac{\partial \phi}{\partial x^i} \left(x^1(t), x^2(t)\right)$$

and

$$\begin{aligned}
\gamma''(t) &= \sum_{i=1}^{2} \ddot{x}^i(t) \frac{\partial \phi}{\partial x^i} \left(x^1(t), x^2(t)\right) + \sum_{i,j=1}^{2} \dot{x}^i(t)\dot{x}^j(t) \frac{\partial^2}{\partial x^i \partial x^j} \left(x^1(t), x^2(t)\right) \\
&= \sum_{k=1}^{2} \left(\ddot{x}^k(t) \frac{\partial \phi}{\partial x^k} + \sum_{i,j=1}^{2} \dot{x}^i(t)\dot{x}^j(t)\Gamma_{ij}^k \left(x^1(t), x^2(t)\right) \frac{\partial \phi}{\partial x^k} \right) + \sum_{i,j=1}^{2} \dot{x}^i(t)\dot{x}^j(t) A_{ij} N.
\end{aligned}$$

Hence, γ is a geodesic if and only if

$$\sum_{k=1}^{2} \left(\ddot{x}^k(t) + \sum_{i,j=1}^{2} \dot{x}^i(t)\dot{x}^j(t)\Gamma_{ij}^k \left(x^1(t), x^2(t)\right) \right) \frac{\partial \phi}{\partial x^k} = 0.$$

Therefore, since the vectors $\left\{\frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}\right\}$ are linearly independent it follows that equations (4.1.1) hold. $\square$

Remark 4.1.6. *Equations* (4.1.1) *are referred to as the geodesic equations. In particular, it follows that geodesic curves that intersect tangentially coincide due to 2.4.9.*

Corollary 4.1.7. *Local isometries map geodesics to geodesics.*

Proof. If $F : S \to \tilde{S}$ is a local isometry, then, shrinking U if necessary, $F \circ \phi : U \to \tilde{S}$ is a chart for $\tilde{S}$. Therefore, as $\tilde{\Gamma}_{ij}^k(u,v) = \Gamma_{ij}^k(u,v)$ for all $(u,v) \in U$, the geodesic equations hold in $\tilde{S}$. $\square$

Exercise 4.1.8. *For a surface S let $\phi : U \to S$ be a chart with first fundamental form*

$$M = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Show that a curve $\gamma : (-\epsilon, \epsilon) \to S$ given by $\gamma(t) = \phi(u(t), v(t))$ is a geodesic if and only if

$$\begin{cases} (Eu' + Fv')' = \frac{1}{2}\left(E_u (u')^2 + 2F_u (u'v') + G_u (v')^2\right) \\ (Fu' + Gv')' = \frac{1}{2}\left(E_v (u')^2 + 2F_v (u'v') + G_v (v')^2\right). \end{cases}$$

Example 4.1.9. *Let $C = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\right\}$. Recall, from statement 2 of Example 3.12.5 that C is isometric to the plane $\Pi = \left\{(x,y,z) \in \mathbb{R}^3 : z = 0\right\}$ through the isometry*

$$F(x,y,0) = (\cos(x), \sin(x), y).$$

From statement 1 of Example 4.1.4 the geodesics of Π are of the form $\gamma(t) = (at+b, ct+d, 0)$, with $a^2+c^2 = 1$ ensuring the geodesic is parameterised by arc length. Thus, using Corollary 4.1.7 it follows that the geodesics of C are of the form

$$\gamma(t) = F(at+b, ct+d) = (\cos(at+b), \sin(at+b), ct+d).$$

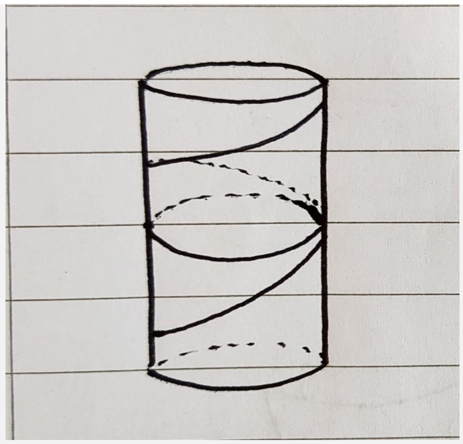


Figure 4.1.2

4.2 Minimising Arc Length

Example 4.2.1. *Note that geodesics do not necessarily minimise the arc length between points. Consider* $\mathbb{S}^2$*, then distinct non-antipodal points can be connected by a small arc and a longer arc along the equator. Both these arcs are geodesics, statement 2 of Example 4.1.4, however, one does not minimise the arc length between the points.*

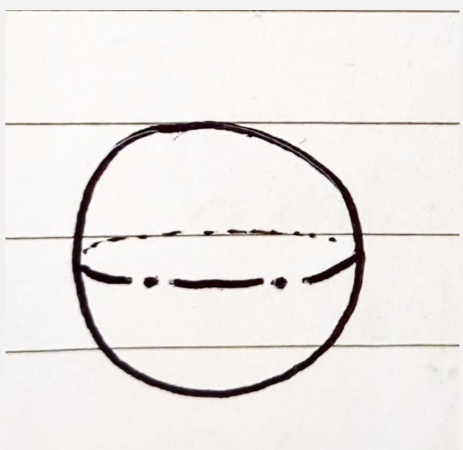


Figure 4.2.1

Definition 4.2.2. *Let* $S \subseteq \mathbb{R}^3$ *be a regular surface, and let* $\gamma : [0, L] \to S$ *be a regular curve. A variation of* γ *is a smooth map* $A : [0, L] \times (-\epsilon, \epsilon) \to S$ *such that the following hold.*

1. $A(t, 0) = \gamma(t)$ *for all* $t \in [0, L]$.
2. $A(0, s) = \gamma(0)$ *for all* $s \in (-\epsilon, \epsilon)$.
3. $A(L, s) = \gamma(L)$ *for all* $s \in (-\epsilon, \epsilon)$.

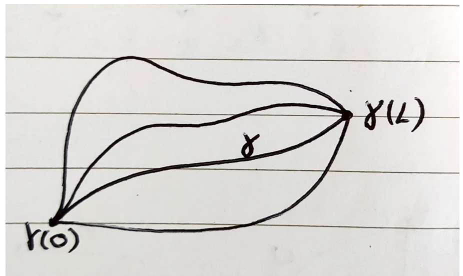


Figure 4.2.2

Proposition 4.2.3. *Let $S \subseteq \mathbb{R}^3$ be a regular surface. Suppose $\gamma : [0, L] \to S$ is a regular curve parameterised by arc length such that if $\tilde{\gamma} : [a, b] \to S$ is a regular curve with $\tilde{\gamma}(a) = \gamma(0)$ and $\tilde{\gamma}(b) = \gamma(L)$ then $L(\gamma) \leq L(\tilde{\gamma})$. Then γ is a geodesic.*

Proof. Let $A : [0, L] \times (-\epsilon, \epsilon) \to S$ be a variation of γ, with $\gamma_s : [0, L] \to S$ given by $\gamma_s(t) = A(t, s)$. As $s \mapsto L(\gamma_s)$ is minimised at $s = 0$, it follows that the derivative of this map is zero at $s = 0$. Note that,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} L(\gamma_s)\Big|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s} \int_0^L |\gamma_s'(t)| \,\mathrm{d}t\Big|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s} \int_0^L \sqrt{\langle \gamma_s'(t), \gamma_s'(t) \rangle} \,\mathrm{d}t\Big|_{s=0} \\
&\overset{(1)}{=} \int_0^L \frac{\mathrm{d}}{\mathrm{d}s} \sqrt{\langle \gamma_s'(t), \gamma_s'(t) \rangle}\Big|_{s=0} \mathrm{d}t \\
&= \int_0^L \frac{\frac{\mathrm{d}}{\mathrm{d}s} \langle \gamma_s'(t), \gamma_s'(t) \rangle \big|_{s=0}}{2\sqrt{\langle \gamma_s'(t), \gamma_s'(t) \rangle}\big|_{s=0}} , \mathrm{d}t \\
&\overset{(2)}{=} \int_0^L \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} \langle \gamma_s'(t), \gamma_s'(t) \rangle \Big|_{s=0}, \mathrm{d}t \\
&= \int_0^L \left\langle \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}t} \right) \Big|_{s=0} \frac{\mathrm{d}\gamma_0(t)}{\mathrm{d}t} \right\rangle \mathrm{d}t \\
&= \int_0^L \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s} \right) \Big|_{s=0} \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} \right\rangle \mathrm{d}t \\
&= \left[\left\langle \frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\Big|_{s=0}, \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} \right\rangle \right]_{t=0}^{t=L} - \int_0^L \left\langle \frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\Big|_{s=0}, \gamma''(t) \right\rangle \mathrm{d}t \\
&\overset{(3)}{=} -\int_0^L \left\langle \frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\Big|_{s=0}, \gamma''(t) \right\rangle \mathrm{d}t \\
&= -\int_0^L \left\langle \frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\Big|_{s=0}, k_n(t) N(\gamma(t)) + k_g(t) \left(N(\gamma(t)) \times \gamma'(t) \right) \right\rangle \mathrm{d}t \\
&\overset{(4)}{=} -\int_0^L k_g(t) \left\langle \frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\Big|_{s=0}, N(\gamma(t)) \times \gamma'(t) \right\rangle \mathrm{d}t. \qquad (4.2.1)
\end{aligned}$$

Equality (1) follows as $s \mapsto \left\langle \sqrt{\langle \gamma_s'(t), \gamma_s'(t) \rangle} \right\rangle$ is smooth in s and t, so that the order or integration can be interchanged. Equality (2) is because γ is parameterised by arc length. Equality (3) is because $\gamma_s(0) = \gamma(0)$, $\gamma_s(L) = \gamma(L)$ and $\frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\Big|_{s=0}$ is zero for $t = 0, L$. Equality (4) is because $\frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\Big|_{s=0}$ belongs to the tangent space of $\gamma(t)$ at $\gamma_s(t) \in S$ for all s and t. Therefore,

$$\int_0^L k_g(t) \left\langle \frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\Big|_{s=0}, N(\gamma(t)) \times \gamma'(t) \right\rangle \mathrm{d}t = 0 \qquad (4.2.2)$$

for any variation A of γ. Suppose that there exists $t_0 \in (0, L)$ such that $k_g(t_0) > 0$, with the case $k_g(t_0) < 0$ being similar. Since k_g is continuous there exists a $\delta > 0$ such that $k_g(t) > 0$ for all $t \in (t_0 - \delta, t_0 + \delta) \subseteq (0, L)$. Let $\phi : U \to S$ be a chart around $\gamma(t_0)$ with $\gamma((t_0 - \delta, t_0 + \delta)) \subseteq \phi(U)$, shrinking δ if necessary. Write $\gamma(t) = \phi(u(t), v(t))$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Note that given any $V : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}^3$ such that $V(t) \in T_{\gamma(t)}S$ for all $t \in (t_0 - \delta, t_0 + \delta)$ there exists a variation A of γ such that

$$V(t) = \frac{\mathrm{d}A(s,t)}{\mathrm{d}s}\bigg|_{s=0}.$$

Indeed, let $V(t) = V_u(t)\frac{\partial \phi}{\partial u} + V_v(t)\frac{\partial \phi}{\partial v}$ and $A(t, s) = \phi\left(u(t) + sV_u(t), v(t) + sV_v(t)\right)$. Then,

$$\frac{\mathrm{d}A(s,t)}{\mathrm{d}s}\bigg|_{s=0} = V(t).$$

So take $V(t) = f(t)\left(N \times \gamma'\right)(t)$ where f is smooth such that $f(t_0) = 1$, $f(t) \leq 1$ and $f(t) = 0$ for $t \in [0, L] \setminus (t_0 - \delta, t_0 + \delta)$. Then,

$$0 = \int_0^L k_g(t)\left\langle \frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\bigg|_{s=0}, \left(N \times \gamma'\right)(t)\right\rangle \mathrm{d}t = \int_{t_0-\delta}^{t_0+\delta} k_g(t)f(t)\,\mathrm{d}t > 0$$

which contradicts (4.2.2). Therefore, $k_g \equiv 0$ and thus a geodesic. $\square$

Example 4.2.4. *Note there may exist points on a regular surface for which no connecting geodesic exists. Consider a plane punctured at a point, then there exist distinct points that cannot be connected by a straight line since the straight line traverses the removed point.*

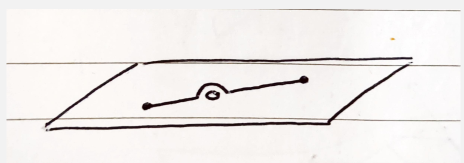


Figure 4.2.3

4.3 Solution to Exercises

Exercise 4.1.8

Solution. Recall that γ is a geodesic if and only if $\langle \gamma''(t), \phi_u \rangle = 0$ and $\langle \gamma''(t), \phi_v \rangle = 0$. Note that

$$\gamma'(t) = \phi_u u'(t) + \phi_v v'(t).$$

Using

$$\langle \gamma''(t), \phi_u \rangle = \left(\langle \gamma'(t), \phi_u \rangle\right)' - \left\langle \gamma'(t), (\phi_u)' \right\rangle$$

it follows that

$$\begin{aligned} 0 &= \left(\langle \phi_u u' + \phi_v v', \phi_u \rangle\right)' - \langle \phi_u u' + \phi_v v', \phi_{uu} u' + \phi_{uv} v' \rangle \\ &= \left(Eu' + Fv'\right)' - \left(\langle \phi_u, \phi_{uu} \rangle \left(u'\right)^2 + \left(\langle \phi_u, \phi_{uv} \rangle + \langle \phi_v, \phi_{uu} \rangle\right)\left(u'v'\right) + \langle \phi_v, \phi_{vv} \rangle \left(v'\right)^2\right). \end{aligned} \tag{4.3.1}$$

Note that

$$\begin{cases} \langle \phi_u, \phi_{uu} \rangle = \frac{1}{2}\frac{\partial}{\partial u}\langle \phi_u, \phi_u \rangle = \frac{1}{2}E_u \\ \langle \phi_u, \phi_{uv} \rangle + \langle \phi_v, \phi_{uu} \rangle = \frac{\partial}{\partial u}\langle \phi_u, \phi_v \rangle = F_u \\ \langle \phi_v, \phi_{uv} \rangle = \frac{1}{2}\frac{\partial}{\partial u}\langle \phi_v, \phi_v \rangle = \frac{1}{2}G_u. \end{cases}$$

Therefore, (4.3.1) becomes

$$0 = (Eu' + Fv')' - \frac{1}{2}\left(E_u\,(u')^2 + 2F_u\,(u'v') + G_u\,(v')^2\right).$$

Similarly, using

$$\langle \gamma''(t), \phi_v \rangle = (\langle \gamma'(t), \phi_v \rangle)' - \left\langle \gamma'(t), (\phi_v)' \right\rangle$$

it follows that

$$0 = (Fu' + Gv')' - \frac{1}{2}\left(E_v\,(u')^2 + 2F_v\,(u'v') + G_v\,(v')^2\right).$$

$\square$

5 Gauss-Bonnet Theorem

5.1 The Local Gauss-Bonnet Theorem

Definition 5.1.1. *A regular surface with boundary $S \subseteq \mathbb{R}^3$ is such that for all $p \in S$ one of the following occurs.*

1. *There is a chart $\phi : U \to S$ for S at p, as per Definition 3.1.1.*
2. *There is an open neighbourhood $U \subseteq \mathbb{R}^2$ of $(0,0)$, an open neighbourhood $V \subseteq \mathbb{R}^3$ of p, and a smooth map $\phi : U \to V$ such that the following hold.*
   - $\phi(0,0) = p$.
   - $\phi : \{(x,y) \in U : y \geq 0\} \to V \cap S$ *is a homeomorphism.*
   - *For every $q \in U$, the differential $\mathrm{d}\phi_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.*

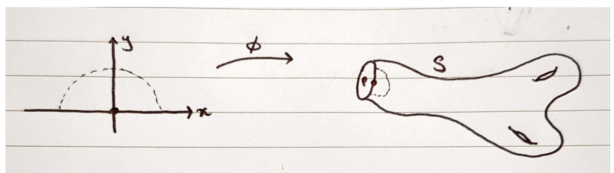


Figure 5.1.1: A chart at a boundary point of S.

Definition 5.1.2. *Let S be a regular surface with a boundary.*

1. *Points $p \in S$ satisfying statement 1 of Definition 5.1.1 are referred to as interior points.*
2. *Points $p \in S$ satisfying statement 2 of Definition 5.1.1 are referred to as boundary points. The set of boundary points of S is denoted ∂S.*

Remark 5.1.3.

1. *Statement 1 of Definition 5.1.1 says that locally at p the subset S looks like a regular surface.*
2. *The smooth map ϕ of statement 2 of Definition 5.1.1 is still referred to as a chart.*
3. *Note that a regular surface is a regular surface with a boundary, just that in this case the boundary is empty.*
4. *Note that ∂S in the sense of statement 2 of Definition 5.1.2 is distinct from ∂S in the sense of topology. In the sense of statement 2 of Definition 5.1.2, ∂S is a union of regular curves. On the other hand, the topological boundary of the set $S \subseteq \mathbb{R}^3$ is S.*

Definition 5.1.4. *Let S be a regular surface with boundary S. For $p \in \partial S$ let*

$$T_pS := \left\{\gamma'(0) \in \mathbb{R}^3 : \gamma : [0,\epsilon) \to S \text{ or } \gamma : (-\epsilon, 0] \to S \text{ smooth with } \gamma(0) = p\right\}.$$

Remark 5.1.5. *For $p \in \partial S$ with ϕ a chart in the sense of statement 2 of Definition 5.1.1, we have $T_pS = \mathrm{d}\phi_{(0,0)}\left(\mathbb{R}^2\right)$.*

An orientation of a regular surface with boundary S induces an orientation on ∂S. If $N : S \to \mathrm{S}^2$ is a Gauss map for S and $\gamma : [a,b] \to \partial S$ is a regular curve, then γ is positively oriented if $N \times \gamma'$ points into S. Therefore, one can consider integrals on regular surfaces with boundary. In particular, if ∂S has a single connected component let $\gamma : [a,b] \to S$ be a positively oriented arc length parameterisation of ∂S. Then for $f : \partial S \to \mathbb{R}$ let

$$\int_{\partial S} f := \int_a^b f(\gamma(t))\,\mathrm{d}t.$$

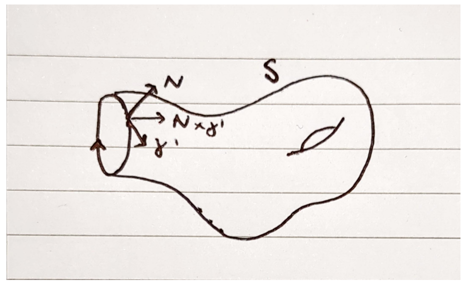


Figure 5.1.2: Orienting the boundary of an oriented surface.

Theorem 5.1.6 (Green's). *Let $F = (F^u, F^v)$ be a real vector field on $\mathbb{R}^2$. Then*

$$\int_{\partial U} F \cdot \mathrm{d}S = \int_U \operatorname{curl}(F)\,\mathrm{d}u\,\mathrm{d}v.$$

Theorem 5.1.7 (Local Gauss-Bonnet). *Let S be a regular surface with a boundary such that there is a chart $\phi : U \to S$ satisfying the following.*

1. *$\bar{U} \subseteq \mathbb{R}^2$ is diffeomorphic to a closed disc, and ϕ is smooth on a neighbourhood of $\bar{U} \subseteq \mathbb{R}^2$.*
2. *$S = \phi\left(\bar{U}\right)$ with $\partial S = \phi\left(\partial U\right)$, where ∂S is as in statement 2 of Definition 5.1.2 and ∂U is the topological boundary of $U \subseteq \mathbb{R}^2$.*

Then

$$\int_{\partial S} k_g\,\mathrm{d}S + \int_S K\,\mathrm{d}A = 2\pi,$$

where k_g is the geodesic curvature of ∂S, and K is the Gauss curvature of S.

Proof. Let $E_1 : \bar{U} \to \mathbb{R}^3$ be given by

$$E_1(u,v) = \frac{\phi_u(u,v)}{|\phi_u(u,v)|}$$

and $E_2 : \bar{U} \to \mathbb{R}^3$ be given by

$$E_2(u,v) = N(\phi(u,v)) \times E_1(u,v).$$

Then for every $(u,v) \in \bar{U}$ the set $\{E_1(u,v), E_2(u,v)\}$ is a basis for $T_{\phi(u,v)}S$. Moreover, $\{E_1, E_2, N\}$ forms an orthonormal basis for $\mathbb{R}^3$ at each point. Let $\gamma : [0,L] \to \partial S$ be a parameterisation of ∂S by arc length, with positive orientation. Then there is a regular curve $\sigma : [0,L] \to \partial U$ such that $\gamma = \phi \circ \sigma$. Moreover, there is a smooth function $\theta : [0,L] \to \mathbb{R}$ such that

$$\gamma'(t) = \cos(\theta(t))E_1(\sigma(t)) + \sin(\theta(t))E_2(\sigma(t)) \in T_{\gamma(t)}S. \tag{5.1.1}$$

Step 1: For every $t \in [0,L]$, we have $k_g(\gamma(t)) = \theta'(t) - \left\langle E_1 \circ \sigma(t), (E_2 \circ \sigma)'(t) \right\rangle.$

Differentiating (5.1.1) with respect to t gives

$$\gamma''(t) = \theta'(t)\left(-\sin(\theta(t))E_1(\sigma(t)) + \cos(\theta(t))E_2(\sigma(t))\right) + \cos(\theta(t))\left(E_1 \circ \sigma\right)'(t) + \sin(\theta(t))\left(E_2 \circ \sigma\right)'(t).$$

Also using (5.1.1) we have

$$\begin{aligned} N(\gamma)\times\gamma' &= N(\gamma)\times(\cos(\theta)\,(E_1\circ\sigma)+\sin(\theta)\,(E_2\circ\sigma)) \\ &= -\sin(\theta)\,(E_1\circ\sigma)+\cos(\theta)\,(E_2\circ\sigma) \end{aligned}$$

Note the relation

$$\langle E_1\circ\sigma, E_1\circ\sigma\rangle = \langle E_2\circ\sigma, E_2\circ\sigma\rangle = 1$$

and

$$\langle E_1\circ\sigma, E_2\circ\sigma\rangle = 0.$$

Differentiating these with respect to t produces the relations

$$\left\langle E_1\circ\sigma, (E_1\circ\sigma)'\right\rangle = \left\langle E_2\circ\sigma, (E_2\circ\sigma)'\right\rangle = 0$$

and

$$\left\langle E_1\circ\sigma, (E_2\circ\sigma)'\right\rangle = -\left\langle (E_1\circ\sigma)', E_2\circ\sigma\right\rangle.$$

Therefore,

$$\begin{aligned} k_g(\gamma(t)) &= \langle\gamma''(t), N(\gamma(t))\times\gamma'(t)\rangle \\ &= \langle\gamma''(t), -\sin(\theta(t))E_1\circ\sigma(t)+\cos(\theta(t))E_2\circ\sigma(t)\rangle \\ &= \theta'(t)+\left\langle\cos(\theta(t))\,(E_1\circ\sigma)'(t)+\sin(\theta(t)\,(E_2\circ\sigma)'(t), -\sin(\theta(t))E_1\circ\sigma(t)+\cos(\theta(t))E_2\circ\sigma(t)\right\rangle \\ &= \theta'(t)+\cos^2(\theta(t))\,\langle(E_1\circ\sigma)'(t), E_2\circ\sigma(t)\rangle-\sin^2(\theta(t))\left\langle(E_2\circ\sigma)'(t), E_1\circ\sigma(t)\right\rangle \\ &= \theta'(t)-\left(\cos^2(\theta)+\sin^2(\theta)\right)\left\langle(E_2\circ\sigma)'(t), E_1\circ\sigma(t)\right\rangle \\ &= \theta'(t)-\left\langle E_1\circ\sigma(t), (E_2\circ\sigma)'(t)\right\rangle. \end{aligned}$$

Step 2: $\int_0^L \left\langle E_1\circ\sigma(t), (E_2\circ\sigma)'(t)\right\rangle \mathrm{d}t = \int_U (K\circ\phi)|\phi_u\times\phi_v|\,\mathrm{d}u\,\mathrm{d}v.$

Note that

$$\left\langle E_1, \frac{\partial E_1}{\partial u}\right\rangle = \frac{1}{2}\frac{\partial}{\partial u}\langle E_1, E_1\rangle = 0$$

and

$$\left\langle E_1, \frac{\partial E_1}{\partial v}\right\rangle = \left\langle E_2, \frac{\partial E_2}{\partial u}\right\rangle = \left\langle E_2, \frac{\partial E_2}{\partial v}\right\rangle = 0.$$

Moreover,

$$\begin{aligned} 0 &= \frac{\partial}{\partial u}\langle E_1, N\circ\phi\rangle \\ &= \left\langle\frac{\partial E_1}{\partial u}, N\circ\phi\right\rangle+\left\langle E_1, \frac{\partial N\circ\phi}{\partial u}\right\rangle \\ &= \left\langle\frac{\partial E_1}{\partial u}, N\circ\phi\right\rangle+\left\langle E_1, \mathrm{d}N_\phi\left(\frac{\partial\phi}{\partial u}\right)\right\rangle \\ &= \left\langle\frac{\partial E_1}{\partial u}, N\circ\phi\right\rangle-A\left(E_1, \frac{\partial\phi}{\partial u}\right). \end{aligned}$$

Therefore, there exists a real function $a: U\to\mathbb{R}$ such that

$$\begin{aligned} \frac{\partial E_1}{\partial u} &= 0E_1+aE_2+\left\langle\frac{\partial E_1}{\partial u}, N\circ\phi\right\rangle N\circ\phi \\ &= aE_2+A\left(E_1, \frac{\partial\phi}{\partial u}\right)N\circ\phi. \end{aligned}$$

Similarly, there exists a real function b such that

$$\frac{\partial E_1}{\partial v} = bE_2+A\left(E_1, \frac{\partial\phi}{\partial v}\right)N\circ\phi,$$

a real function c such that

$$\frac{\partial E_2}{\partial u} = cE_1 + A\left(E_2, \frac{\partial \phi}{\partial u}\right) N \circ \phi,$$

and a real function d such that d such that

$$\frac{\partial E_2}{\partial v} = dE_1 + A\left(E_2, \frac{\partial \phi}{\partial v}\right) N \circ \phi.$$

For $i, j = 1, 2$ let the real functions c_{ij} be such that

$$\frac{\partial \phi}{\partial u} = c_{11}E_1 + c_{12}E_2$$

and

$$\frac{\partial \phi}{\partial v} = c_{21}E_1 + c_{22}E_2.$$

Then

$$\begin{aligned}
\left\langle \frac{\partial E_1}{\partial u}, \frac{\partial E_2}{\partial v} \right\rangle - \left\langle \frac{\partial E_2}{\partial u}, \frac{\partial E_1}{\partial v} \right\rangle &= A\left(E_1, \frac{\partial \phi}{\partial u}\right) A\left(E_2, \frac{\partial \phi}{\partial v}\right) - A\left(E_2, \frac{\partial \phi}{\partial u}\right) A\left(E_1, \frac{\partial \phi}{\partial v}\right) \\
&= \left(c_{11}A(E_1, E_1) + c_{12}A(E_1, E_2)\right)\left(c_{21}A(E_2, E_1) + c_{22}A(E_2, E_2)\right) \\
&\quad - \left(c_{11}A(E_2, E_1) + c_{12}A(E_2, E_2)\right)\left(c_{21}A(E_1, E_1) + c_{22}A(E_1, E_2)\right) \\
&= (c_{11}c_{22} - c_{12}c_{21})(A(E_1, E_1)A(E_2, E_2) - A(E_1, E_2)A(E_2, E_1)) \\
&= \left|\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}\right| \det(\Sigma) \\
&= \left|\phi_u \times \phi_v\right| K \circ \phi,
\end{aligned}$$

where we have used the fact that the first fundamental form M in the basis $\{E_1, E_2\}$ is the identity matrix and so has a determinant one. Under a change of basis, P, the first and second fundamental forms change to $P^\top MP$ and $P^\top \Sigma P$ respectively. Thus, the quantity $\frac{\det(\Sigma)}{\det(M)}$ is independent of the choice of basis. Now let $\sigma(t) = (u(t), v(t))$, where $u(t)$ and $v(t)$ are some smooth functions. Then

$$\begin{aligned}
\frac{\partial}{\partial t}(E_2 \circ \sigma)(t) &= \frac{\partial}{\partial t}(E_2(u(t), v(t))) \\
&= \frac{\partial E_2}{\partial u}(u(t), v(t))v'(t) + \frac{\partial E_2}{\partial v}(u(t), v(t))v'(t).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^L \langle E_1 \circ \sigma(t), (E_2 \circ \sigma)'(t) \rangle \, \mathrm{d}t &= \int_0^L \left\langle E_1(u(t), v(t)), \frac{\partial E_2}{\partial u}(u(t), v(t)) \right\rangle u'(t) \, \mathrm{d}t \\
&\quad + \int_0^L \left\langle E_1(u(t), v(t)), \frac{\partial E_2}{\partial v}(u(t), v(t)) \right\rangle v'(t) \, \mathrm{d}t \\
&= \int_{\partial U} \left\langle E_1(u, v), \frac{\partial E_2}{\partial u}(u, v) \right\rangle \mathrm{d}u + \int_{\partial U} \left\langle E_1(u, v), \frac{\partial E_2}{\partial u}(u, v) \right\rangle \mathrm{d}v \\
&\overset{(1)}{=} \int_U \frac{\partial}{\partial u}\left(\left\langle E_1(u, v), \frac{\partial E_2}{\partial v}(u, v) \right\rangle\right) - \frac{\partial}{\partial v}\left(\left\langle E_1(u, v), \frac{\partial E_2}{\partial u}(u, v) \right\rangle\right) \mathrm{d}u \, \mathrm{d}v \\
&= \int_U \left(\left\langle \frac{\partial E_1}{\partial u}, \frac{\partial E_2}{\partial v} \right\rangle + \left\langle E_1, \frac{\partial^2 E_2}{\partial u \partial v} \right\rangle\right) - \left(\left\langle \frac{\partial E_1}{\partial v}, \frac{\partial E_2}{\partial v} \right\rangle + \left\langle E_1, \frac{\partial^2 E_2}{\partial v \partial u} \right\rangle\right) \mathrm{d}u \, \mathrm{d}v \\
&= \int_U \left\langle \frac{\partial E_1}{\partial u}, \frac{\partial E_2}{\partial v} \right\rangle - \left\langle \frac{\partial E_1}{\partial v}, \frac{\partial E_2}{\partial v} \right\rangle \mathrm{d}u \, \mathrm{d}v \\
&= \int_U |\phi_u \times \phi_v| K \circ \phi \, \mathrm{d}u \, \mathrm{d}v,
\end{aligned}$$

where in (1) we have applied Theorem 5.1.6 by noting that

$$\int_{\partial U} F \cdot \mathrm{d}S = \int_0^L F^u u'(t) + F^v v'(t)\,\mathrm{d}t$$

and

$$\mathrm{curl}(F) = \partial_u F^v - \partial_v F^u.$$

Step 3: $\int_0^L \theta'(s)\,\mathrm{d}s = 2\pi$.
By the fundamental theorem of calculus,

$$\int_0^L \theta'(s)\,\mathrm{d}s = \theta(L) - \theta(0). \tag{5.1.2}$$

For a closed curve γ we have $\gamma'(0) = \gamma'(L)$, and thus from (5.1.1) we have that $\theta(L) - \theta(0)$ is a multiple of 2π. By assumption, we have a diffeomorphism $\Psi : \overline{B(0,1)} \to \bar{U}$, which can be used to construct a family of regular curves $\left(\sigma_s : [0,L] \to \bar{U}\right)_{s\in[0,1]}$ such that $\sigma_0 \equiv \sigma$ and σ_1 is a circle of radius ϵ in U. Then the continuous family of curves $(\gamma_s)_{s\in[0,1]}$, where $\gamma_s := \phi \circ \sigma_s$, can be considered as a continuous perturbation of γ to some small closed curve in S. Note that the left-hand side of (5.1.2) depends continuously on the curves γ_s. Since the value of the integral of the left-hand side is an integer multiple of 2π for all γ_s, it must be the case that the value of the left-hand side for γ_s is equal to the value of the left-hand side for γ_1. Since the curve σ_1 is small, the map E_1 on σ_1 is essentially constant, since E is smooth. Observe that $\theta_1(t)$ is the angle between $\gamma_1'(t)$ and $E_1 \circ \sigma_1(t)$. Since, γ_1 revolves counter-clockwise around a small circle, and $E_1 \circ \sigma$ is near constant on that circle, the change in angle is 2π.
Step 4: Conclude.
Using step 1, step 2 and step 3 we have

$$\begin{aligned}
\int_{\partial S} k_g\,\mathrm{d}s + \int_S K\,\mathrm{d}A &= \int_0^L k_g(\gamma(t))\,\mathrm{d}t + \int_U (K\circ\phi)|\phi_u \times \phi_v|\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^L \theta'(t)\,\mathrm{d}t - \int_0^L \left\langle E_1\circ\sigma(t), (E_2\circ\sigma)'(t)\right\rangle\,\mathrm{d}t + \int_U (K\circ\phi)|\phi_u\times\phi_v|\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^L \theta'(t)\,\mathrm{d}t \\
&= 2\pi.
\end{aligned}$$

□

Remark 5.1.8. *Theorem 5.1.7 links curvature, an intrinsic quantity, and Gauss curvature, an extrinsic quantity, to a topological quantity.*

Exercise 5.1.9. *For a smooth function f let*

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : (x,y,f(x,y))\right\}.$$

Suppose that $f(x,y) = 0$ for $x^2 + y^2 \geq R^2$ for some $R > 0$. If $K \geq 0$, show that $K \equiv 0$.

5.2 The n-gon Gauss Bonnet

Definition 5.2.1. *Given $n \geq 1$, a continuous map $\beta : [0,L] \to \mathbb{R}^2$ is a curvilinear n-gon if the following hold.*

1. *$\beta(0) = \beta(L)$.*
2. *$\beta|_{[0,L)}$ is injective.*

3. There exists a subdivision $0 = t_0 < \cdots < t_n = L$ such that the following hold.

   (a) $\beta|_{(t_i, t_{i+1})}$ is a regular curve for $i = 0, \ldots, n-1$.

   (b) $\beta'\left(t_i^-\right) = \lim_{t \nearrow t_i} \beta'(t)$ exists for $i = 1, \ldots, n$.

   (c) $\beta'\left(t_i^+\right) = \lim_{t \searrow t_i} \beta'(t)$ exists for $i = 0, \ldots, n-1$.

   (d) Each pair $\left(\beta'\left(t_0^+\right), \beta'\left(t_n^-\right)\right), \left(\beta'\left(t_1^-\right), \beta'\left(t_1^+\right)\right), \ldots, \left(\beta'\left(t_{n-1}^-\right), \beta'\left(t_{n-1}^+\right)\right)$ is linearly independent.

Remark 5.2.2.

1. *A curvilinear 3-gon is referred to as a curvilinear triangle.*
2. *The points $\beta(t_i)$ are referred to as the vertices and the curves $\beta\left((t_i, t_{i+1})\right)$ are referred to as the edges of the curvilinear n-gon.*

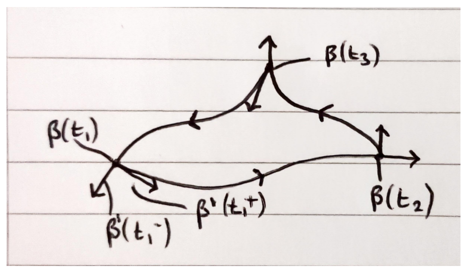


Figure 5.2.1: A curvilinear triangle.

If S is a regular surface with chart $\phi : U \to S$, and $\beta \subseteq U$ is a curvilinear n-gon, then $\phi(\beta)$ is referred to as a curvilinear n-gon in S. We assume that the linear independence of the vectors of statement 3(d) of Definition 5.2.1 is preserved.

Definition 5.2.3. *Let S be a regular surface with chart $\phi : U \to S$. Let β be a curvilinear n-gon and set $\gamma := \phi \circ \beta$. Then the exterior angle θ_i for $i = 1, \ldots, n$, between $\gamma'\left(t_i^-\right)$ and $\gamma'\left(t_i^+\right)$ is $\theta_i \in (-\pi, \pi)$ such that*

$$\cos(\theta_i) = \frac{\left\langle \gamma'\left(t_i^-\right), \gamma'\left(t_i^+\right) \right\rangle}{\left|\gamma'\left(t_i^-\right)\right| \left|\gamma'\left(t_i^+\right)\right|}$$

with

- *$\theta_i > 0$ if $\left\{\gamma'\left(t_i^-\right), \gamma'\left(t_i^+\right), N\right\}$ is a positively oriented basis of $\mathbb{R}^3$, and*
- *$\theta_i < 0$ if $\left\{\gamma'\left(t_i^-\right), \gamma'\left(t_i^+\right), N\right\}$ is a negatively oriented basis of $\mathbb{R}^3$,*

where N is a unit normal vector to S.

Remark 5.2.4.

1. *Note suitable adjustments are to be made to Definition 5.2.3 when $i \in \{0, n\}$.*
2. *The interior angle is given by $\alpha_i = \pi - \theta_i$.*

Theorem 5.2.5 (n-gon Gauss-Bonnet)**.** *Let $S' \subseteq \mathbb{R}^3$ be a regular oriented surface. Suppose $S \subseteq S'$ is a regular surface with boundary such that ∂S is a curvilinear n-gon, with edges whose arc length parameterisations are*

$\gamma_i = \gamma|_{(t_{i-1},t_i)}$ *for* $i = 1, \ldots, n$ *and meet at exterior angles* θ_i*. Then,*

$$\sum_{i=1}^{n} \int_{\gamma_i} k_g \, \mathrm{d}S + \sum_{i=1}^{n} \theta_i + \int_S K \, \mathrm{d}A = 2\pi.$$

Proof. Recall the proof of Theorem 5.1.7. The identity of step 1 holds on the intervals (t_{i-1}, t_i) for $i = 1, \ldots, n$. Step 2 remains unchanged. Then using similar arguments as those made in step 3 we deduce that

$$\int_{t_0}^{t_1} \theta'(t) \, \mathrm{d}t + \cdots + \int_{t_{n-1}}^{t_n} \theta'(t) \, \mathrm{d}t = 2\pi - \sum_{i=1}^{n} \theta_i,$$

since γ' still rotates by 2π, however, the left-hand side does not include the rotations that occur at the vertices. □

Remark 5.2.6. *In the context of Theorem 5.2.5, if* $n = 3$ *and the edges are geodesics, then*

$$\begin{aligned} \int_S K \, \mathrm{d}A &= 2\pi - (\theta_1 + \theta_2 + \theta_3) \\ &= \alpha_1 + \alpha_2 + \alpha_3 - \pi. \end{aligned}$$

In particular, we consider the following cases.

1. *If* S' *is a plane, then* $\alpha_1 + \alpha_2 + \alpha_3 = \pi$.
2. *If* S' *is a unit sphere, then* $\alpha_1 + \alpha_2 + \alpha_3 = \pi + \operatorname{area}(T)$.
3. *If* $K_{S'} < 0$*, then* $\alpha_1 + \alpha_2 + \alpha_3 < \pi$.

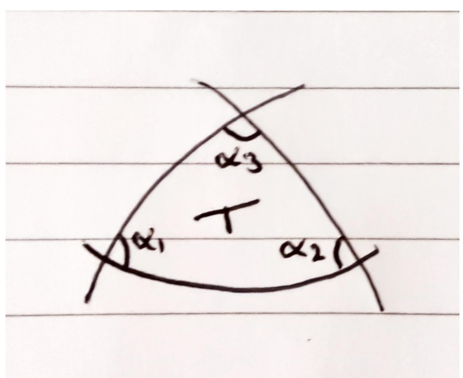


Figure 5.2.2: Curvilinear triangle and its interior angles on a sphere.

Exercise 5.2.7. *Let* $S \subseteq \mathbb{R}^3$ *be a regular surface with* $p \in S$*. Consider a curvilinear triangle* $T \subseteq S$ *whose sides are geodesics, whose interior contains* p*, and whose interior angles are* $\{\alpha_1, \alpha_2, \alpha_3\}$*. Show that*

$$\lim_{T \to p} \frac{\sum_{i=1}^{3} \alpha_i - \pi}{\operatorname{area}(T)} \to K(p).$$

5.3 Triangulation and the Euler Characteristic

Definition 5.3.1. *Let $S \subseteq \mathbb{R}^3$ be a compact oriented surface, with or without boundary. Then a triangulation of S is a collection $\{T_1, \ldots, T_n\}$ of curvilinear triangles in S such that the following hold.*

1. $\bigcup_{i=1}^n T_i = S$.
2. *If $i \neq j$ and $T_i \cap T_j \neq \emptyset$ then $T_i \cap T_j$ is either a vertex or an edge.*
3. *For any edge in the interior of S, exactly two triangles share the edge.*
4. *For any edge in ∂S, there is a unique triangle containing the edge.*

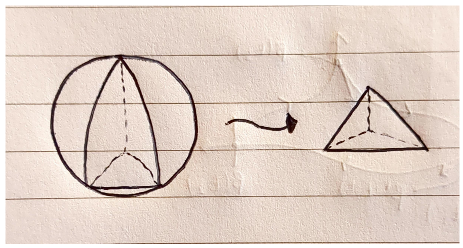


Figure 5.3.1: Triangulation of $\mathbb{S}^2$.

Remark 5.3.2. *Note how in Definition 5.3.1 we take curvilinear triangles to include the interior of the curves given in Definition 5.2.1.*

Definition 5.3.3. *The Euler characteristic of a triangulation $\bigcup_{i=1}^n T_i$ is*

$$\chi\left(\bigcup_{i=1}^{n} T_i\right) = V - E + F,$$

where

- *E is the number of distinct edges,*
- *V is the number of distinct vertices, and*
- *F is the number of faces.*

Theorem 5.3.4.

1. *Every regular surface, with or without boundary, admits a triangulation.*
2. *If the boundary of each triangle is positively oriented, then adjacent triangles of an edge determine opposite orientations of the edge.*
3. *For triangulations $\bigcup_{i=1}^n T_i$ and $\bigcup_{i=1}^m T_i'$ of a compact oriented surface S, we have that*

$$\chi\left(\bigcup_{i=1}^{n} T_i\right) = \chi\left(\bigcup_{i=1}^{m} T_i'\right).$$

That is, the Euler characteristic of a surface is a topological invariant.

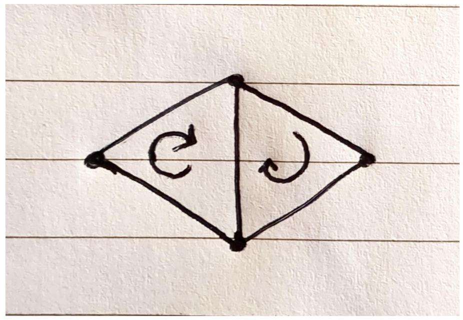


Figure 5.3.2: Orientation of edges.

Remark 5.3.5. *In light of statement 3 of Theorem 5.3.4, for a compact oriented surface S one usually writes $\chi(S)$ for the Euler characteristic of the surface.*

Example 5.3.6.

1. *For S^2, we can use the triangulation of Figure 5.3.1 to deduce that*

$$\chi\left(\mathrm{S}^2\right) = 4 - 6 + 4 = 2$$

as $V = 4$, $E = 6$ and $F = 4$.

2. *For the torus, T^2, we can represent a triangulation using the simplified perspective of Figure 5.3.3 to deduce that*

$$\chi\left(T^2\right) = 4 - 12 + 8 = 0$$

as $V = 4$, $E = 12$ and $F = 8$ after appropriate identification of vertices and edges in Figure 5.3.3.

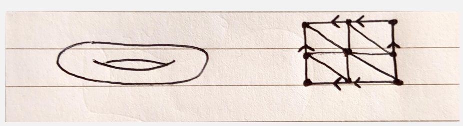


Figure 5.3.3: Triangulation of the torus.

3. *For the cylinder with boundary, C, we use the triangulation of Figure 5.3.4 to deduce that*

$$\chi(C) = 6 - 14 + 8 = 0$$

as $V = 6$, $E = 14$ and $F = 8$.

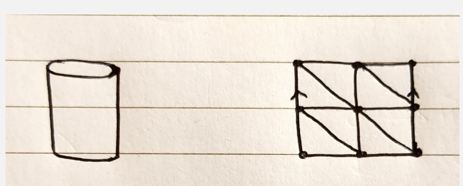


Figure 5.3.4: Triangulation of the cylinder.

4. *Consider a surface Σ_g with g handles as depicted in Figure 5.3.5. Then*

$$\chi(\Sigma_g) = 2 - 2g.$$

Indeed, note that a triangulation of Σ_g can be constructed from a triangulation of Σ_{g-1} and Σ_1 by identifying triangles from each surface, Figure 5.3.5. For this triangulation observe that

$$\begin{cases} V_{\Sigma_g} = V_{\Sigma_{g-1}} + V_{\Sigma_1} - 3 \\ E_{\Sigma_g} = E_{\Sigma_{g-1}} + E_{\Sigma_1} - 3 \\ F_{\Sigma_g} = F_{\Sigma_{g-1}} + F_{\Sigma_1} - 2. \end{cases}$$

So suppose that $\chi(\Sigma_{g-1}) = 2 - 2(g-1)$ and note that $\Sigma_1 = T^2$ so that $\chi(\Sigma_1) = \chi\left(T^2\right) = 0$. Then it follows that

$$\begin{aligned} \chi(\Sigma_g) &= V_{\Sigma_g} - E_{\Sigma_g} + F_{\Sigma_g} \\ &= \chi(\Sigma_{g-1}) + \chi(\Sigma_1) - 2 \\ &= 2 - 2(g-1) - 2 \\ &= 2 - 2g. \end{aligned}$$

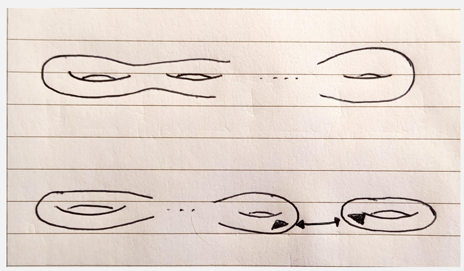


Figure 5.3.5: Triangulation of Σ_g, and the construction of Σ_g from Σ_{g-1} and Σ_1.

Theorem 5.3.7. *Let $S \subseteq \mathbb{R}^3$ be a compact connected surface without boundary. Then,*

$$\chi(S) \in \{2, 0, -2, -4, \dots\}.$$

Furthermore, if $S' \subseteq \mathbb{R}^3$ is another regular surface such that $\chi(S') = \chi(S)$, then S' is homeomorphic to S.

Definition 5.3.8. *If $S \subseteq \mathbb{R}^3$ is a compact connected surface. Then the genus of S is given by*

$$g(S) = \frac{2 - \chi(S)}{2}.$$

Remark 5.3.9.

1. *Note that from Theorem 5.3.7 we have that $g(S) \in \{0, 1, 2, \dots\}$.*
2. *Informally, the genus of a surface measures the number of holes in the surface.*

5.4 Gauss-Bonnet

Theorem 5.4.1. *Let $S \subseteq \mathbb{R}^3$ be a compact surface with or without boundary. Then*

$$\int_{\partial S} k_g \, \mathrm{d}S + \int_S K \, \mathrm{d}A = 2\pi\chi(S)$$

when ∂S is positively oriented. In particular, if $\partial S = \emptyset$ then

$$\int_S K \, \mathrm{d}A = 2\pi\chi(S).$$

Proof. Let $(T_i)_{i=1}^n$ be a triangulation of S. For each $i = 1, \ldots, n$ let θ_{ij} denote the exterior angle of the j^{th} vertex of T_i. Using Theorem 5.2.5 it follows that

$$\begin{aligned}\int_{T_i} K \, \mathrm{d}A + \int_{\partial T_i} k_g \, \mathrm{d}S &= 2\pi - \sum_{j=1}^{3} \theta_{ij} \\ &= \sum_{j=1}^{3} \alpha_{ij} - \pi,\end{aligned}$$

where $\alpha_{ij} = \pi - \theta_{ij}$ is the interior angle. Therefore,

$$\int_S K \, \mathrm{d}A + \sum_{i=1}^{n} \int_{\partial T_i} k_g \, \mathrm{d}S = \sum_{i=1}^{n} \sum_{j=1}^{3} \alpha_{ij} - n\pi. \tag{5.4.1}$$

Since each interior edge appears twice with opposite orientations their contributions cancel, thus

$$\sum_{i=1}^{n} \int_{\partial T_i} k_g \, \mathrm{d}S = \int_{\partial S} k_g \, \mathrm{d}S. \tag{5.4.2}$$

At an interior vertex the α_{ij} sum to 2π. At a boundary, the α_{ij} sum to π. Thus,

$$\sum_{i=1}^{n} \sum_{j=1}^{3} \alpha_{ij} - n\pi = 2\pi V - \pi V_b - \pi F,$$

where V_b is the number of boundary vertices and where we have also used the fact that $n = F$. Observe that

$$\begin{aligned}3F &= 2E_i + E_b \\ &= 2E - E_b \\ &= 2E - V_b,\end{aligned}$$

where E_i is the number of interior edges and E_b is the number of boundary edges. Hence,

$$\begin{aligned}\sum_{i=1}^{n} \sum_{j=1}^{3} \alpha_{ij} - n\pi &= 2\pi V + \pi(3F - 2E) - \pi F \\ &= 2\pi(V - E + F) \\ &= 2\pi\chi(S).\end{aligned} \tag{5.4.3}$$

Using (5.4.2) and (5.4.3) in (5.4.1) it follows that

$$\int_S K \, \mathrm{d}A + \int_{\partial S} k_g \, \mathrm{d}S = 2\pi\chi(S).$$

□

Example 5.4.2. *For $a, b > 0$ consider*

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1\right\},$$

with chart

$$\phi(u,v) = (a\cos(u)\cos(v), a\sin(u)\cos(v), b\sin(v)).$$

Then,

$$\begin{cases} \phi_u(u,v) = (-a\sin(u)\cos(v), a\cos(u)\cos(v), 0) \\ \phi_v(u,v) = (-a\cos(u)\sin(v), -a\sin(u)\sin(v), b\cos(v)) \end{cases}$$

and

$$\begin{cases} \phi_{uu}(u,v) = (-a\cos(u)\cos(v), -a\sin(u)\cos(v), 0) \\ \phi_{uv}(u,v) = (a\sin(u)\sin(v), -a\cos(u)\sin(v), 0) \\ \phi_{vv}(u,v) = (-a\cos(u)\cos(v), -a\sin(u)\cos(v), -b\sin(v)). \end{cases}$$

Therefore,

$$N(\phi(u,v)) = \pm\frac{(b\cos(u)\cos(v), b\sin(u)\cos(v), a\sin(v))}{\sqrt{b^2\cos^2(v) + a^2\sin^2(v)}}.$$

Furthermore,

$$M = \begin{pmatrix} a^2\cos^2(v) & 0 \\ 0 & a^2\sin^2(v) + b^2\cos^2(v) \end{pmatrix}$$

and

$$\Sigma = \frac{1}{\sqrt{a^2\sin^2(v) + b^2\cos^2(v)}} \begin{pmatrix} -ab\cos^2(v) & 0 \\ 0 & -ab \end{pmatrix}.$$

Hence,

$$K(\phi(u,v)) = \frac{b^2}{\left(a^2\sin^2(v) + b^2\cos^2(v)\right)^2}.$$

Therefore, as S is a diffeomorphic to a sphere we have $\chi(S) = 2$, so by Theorem 5.4.1 it follows that

$$\begin{aligned} 4\pi &= \int_S K \,\mathrm{d}A \\ &= \int_0^\pi \int_0^{2\pi} K(\phi(u,v))|\phi_u(u,v) \times \phi_v(u,v)| \,\mathrm{d}u \,\mathrm{d}v \\ &= \int_0^\pi \int_0^{2\pi} \frac{b^2 a|\cos(v)|\sqrt{a^2\sin^2(v) + b^2\cos^2(v)}}{\left(a^2\sin^2(v) + b^2\cos^2(v)\right)^2} \,\mathrm{d}u \,\mathrm{d}v. \end{aligned}$$

Upon simplifications we deduce that

$$\int_0^1 \frac{1}{\left(b^2 + (a^2 - b^2)\,x^2\right)^{\frac{3}{2}}} \,\mathrm{d}x = \frac{1}{ab^2}.$$

Corollary 5.4.3. *Let $S \subseteq \mathbb{R}^3$ be a compact connected oriented surface without boundary and with $K \geq 0$. Then S is homeomorphic to S^2.*

Proof. From Theorem 5.4.1 we have

$$2\pi\chi(S) = \int_S K \,\mathrm{d}S \geq 0.$$

Thus,

$$2\pi(2 - 2g(S)) \geq 0$$

so that $g(S) = 0$ or $g(S) = 1$.

- If $g(S) = 0$ then S is homeomorphic to S^2.
- If $g(S) = 1$ then $\int_S K \,\mathrm{d}S = 0$ which implies that $K \equiv 0$. This contradicts Proposition 3.8.6.

Therefore, S is homeomorphic to S^2. □

Exercise 5.4.4. *Suppose $S \subseteq \mathbb{R}^3$ is a compact, connected and oriented surface without boundary that is not homeomorphic to a sphere. Show that there are points $p_1, p_2, p_3 \in S$ such that $K(p_1) < 0$, $K(p_2) = 0$ and $K(p_3) > 0$, where K is the Gauss curvature of the surface.*

Corollary 5.4.5. *Let $S \subseteq \mathbb{R}^3$ be a compact connected surface without boundary and with $K > 0$. If γ_1 and γ_2 are simple closed geodesics, that is they have no self-intersection except at the endpoints, then γ_1 and γ_2 intersect each other.*

Proof. Suppose that $\mathrm{im}(\gamma_1) \cap \mathrm{im}(\gamma_2) = \emptyset$. From Corollary 5.4.3 we know that S is homeomorphic to S^2. Hence, any simple closed curve divides S into two components. Consequently, γ_1 and γ_2 bound a region $\Sigma \subseteq S$ that is homeomorphic to a cylinder for which the Euler characteristic is zero. Therefore, since $\partial\Sigma = \mathrm{im}(\gamma_1) \cup \mathrm{im}(\gamma_2)$, with γ_1 and γ_2 being geodesics, it follows by Theorem 5.4.1 that

$$\int_\Sigma K \,\mathrm{d}A = 0,$$

which contradicts $K > 0$ on S. □

5.5 Solution to Exercises

Exercise 5.1.9

Solution. Let $\gamma : [0, 2\pi(2R)) \to S$ be given by

$$\gamma(t) = 2R\left(\cos\left(\frac{t}{2R}\right), \sin\left(\frac{t}{2R}\right), 0\right),$$

so that γ bounds

$$D = \left\{(x, y, f(x, y)) : x^2 + y^2 \leq (2R)^2\right\},$$

which is diffeomorphic to a disk. As S is locally a plane along points of γ, the normal curvature at γ is zero, hence the geodesic curvature of γ is given by the curvature which is equal to $\frac{1}{2R}$. Therefore, by Theorem 5.1.7 it follows that

$$\int_{\mathrm{im}(\gamma)} \frac{1}{2R} \,\mathrm{d}S + \int_D K \,\mathrm{d}A = 2\pi,$$

which implies that

$$\int_D K \,\mathrm{d}A = 0.$$

Therefore, since $K \geq 0$ it follows that $K \equiv 0$ on D, and thus $K \equiv 0$ on S. □

Exercise 5.2.7

Solution. By Theorem 5.2.5 we have

$$\sum_{i=1}^{3} \theta_i + \int_T K \,\mathrm{d}A = 2\pi,$$

so that

$$\int_T K\,\mathrm{d}A = \sum_{i=1}^{3} \alpha_i - \pi.$$

Therefore, as K is continuous it follows that

$$\lim_{T \to p} \frac{\sum_{i=1}^{3} \alpha_i - \pi}{\operatorname{area}(T)} = \lim_{T \to p} \frac{\int_T K\,\mathrm{d}A}{\operatorname{area}(T)} = K(p).$$

$\square$

Exercise 5.4.4

Solution. As S is not homeomorphic to S^2 it follows from Corollary 5.4.3 that there exists a $p_1 \in S$ such that $K(p_1) < 0$. As S is a compact surface it follows from Proposition 3.8.6 that there exists a $p_3 \in S$ with $K(p_3) > 0$. Since S is connected there exists a path connecting p_1 and p_3. Since K varies continuously along this path it follows by the intermediate value theorem that there exists a $p_2 \in S$ such that $K(p_2) = 0$. $\square$