Theory of Partial Differential Equations*

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1 Introduction

1.1 Introduction to Partial Differential Equations

1.1.1 Partial Differential Equation

Let $u = u(x_1, \ldots, x_d)$ be a function in d variables, with partial derivatives denoted $\partial_{x_j} u, \partial_{x_i x_j} u, \ldots$, then a partial differential equation (PDE) has the form

$$F\left(x_1, \ldots, x_d, u, \partial_{x_1} u, \ldots, \partial_{x_d} u, \partial_{x_1 x_1} u, \ldots, \partial_{x_d x_d}, \partial_{x_1 x_1 x_1}\right) = 0.$$

We get different types of PDEs depending on the properties of F.

1. If F is linear with respect to u we have a linear PDE, otherwise we have a non-linear PDE.
   (a) The PDE is semi-linear if it is non-linear only with respect to u.
   (b) The PDE is quasi-linear if it is linear with respect to the highest order derivative of u.
   (c) The PDE is fully non-linear if F is non-linear with respect to the highest order derivative of u.
2. The order of the PDE is equal to the highest order of differentiation contained in F.

1.1.2 Linear PDEs

Definition 1.1.1. *An operator L acting on functions u, v is linear if the following hold.*

1. *$L(u+v) = L(u) + L(v)$ for all functions u and v.*
2. *$L(cu) = cL(u)$ for all functions u and $c \in \mathbb{R}$.*

An operator L on functions $u : \mathbb{R} \to \mathbb{R}$ of the form

$$L = \sum_{i=0}^{m} f_i(x) \partial_x^{(i)}$$

where $x \in \mathbb{R}$ and $f_i : \mathbb{R} \to \mathbb{R}$ is linear and is called the general linear differential operator. The order of the operator is m, and we use the notation

$$\partial_x^{(i)} = \partial_{\underbrace{x \ldots x}_{i}}.$$

Theorem 1.1.2 (Superposition Principle)**.** *Let $u_1, \ldots, u_k$ be solutions to the same linear PDE. Then $u = c_1 u_1 + \cdots + c_k u_k$ with $c_1, \ldots, c_k \in \mathbb{R}$ is also a solution to the linear PDE.*

1.2 Calculus Preliminaries

Throughout, a domain will refer to an open connected set and will usually be denoted Ω.

Definition 1.2.1. *A point $\mathbf{x} \in \Omega \subset \mathbb{R}^d$ is a boundary point if for any $r > 0$ we have that $B_r(\mathbf{x}) \cap \Omega \neq \emptyset$ and $B_r(\mathbf{x}) \cap \left(\mathbb{R}^d \setminus \Omega\right) \neq \emptyset$.*

Definition 1.2.2. *For a domain Ω, the set of boundary points is denoted $\partial\Omega$.*

Definition 1.2.3. *A domain Ω is a $\mathcal{C}^1$-domain if the following holds for every $\mathbf{x} \in \partial\Omega$. For $\mathbf{x} \in \partial\Omega$ there exists a system of coordinates*

$$y_1, \ldots, y_{d-1}, y_d = \left(\mathbf{y}', y_d\right),$$

whose origin is $\mathbf{x}$, along with a ball $B(\mathbf{x})$ and a function φ defined on a neighbourhood $\mathcal{N} \subset \mathbb{R}^{d-1}$ of $\mathbf{y}' = \mathbf{0}'$ such that $\varphi \in \mathcal{C}^1(\mathcal{N})$, $\varphi(\mathbf{0}') = 0$. Moreover,

1. $\partial\Omega \cap B(\mathbf{x}) = \{(\mathbf{y}', y_d) : y_d = \varphi(\mathbf{y}'), \mathbf{y}' \in \mathcal{N}\}$, *and*
2. $\Omega \cap B(\mathbf{x}) = \{(\mathbf{y}', y_d) : y_d > \varphi(\mathbf{y}'), \mathbf{y}' \in \mathcal{N}\}$.

In words, this says that at every point on the boundary of a $\mathcal{C}^1$-domain has a well-defined tangent. Moreover, the normal vectors to these tangents vary continuously along the boundary. The pairs $(\varphi, \mathcal{N})$ are called local charts. We can generalize this definition to C^k-domains by requiring the functions φ to be $\mathcal{C}^k$-functions.

Definition 1.2.4. *A domain Ω is smooth if Ω is a $\mathcal{C}^k$-domain for all $k \geq 1$. In such a case we say Ω is a $\mathcal{C}^\infty$-domain.*

If the $\varphi : \Omega \to \mathbb{R}$ are only Lipschitz, say with constant L, then we call Ω a Lipschitz domain.

Integration by Parts Formula

Theorem 1.2.5 (Gauss Divergence Formula)**.** *Let $\Omega \subset \mathbb{R}^d$ be a $\mathcal{C}^1$-domain and let $\mathbf{F} = (F_1, \dots, F_d) : \Omega \to \mathbb{R}^d$ be a vector field with $\mathbf{F} \in \mathcal{C}^1\left(\bar{\Omega}\right)$. Then the following holds*

$$\int_\Omega \nabla \cdot \mathbf{F}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} \mathbf{F}(\sigma) \cdot \mathbf{n}(\sigma)\,\mathrm{d}\sigma,$$

where

- $\nabla \cdot \mathbf{F} = \partial_{x_1} F_1 + \cdots + \partial_{x_d} F_d$ *is the divergence of* $\mathbf{F}$,
- $\mathbf{n}$ *is the unit outward normal vector to* $\partial\Omega$, *and*
- $\mathrm{d}\sigma$ *is the surface measure on* $\partial\Omega$ *given by the local charts*

$$\mathrm{d}\sigma = \sqrt{1 + |\nabla\varphi(\mathbf{y}')|^2}\mathrm{d}\mathbf{y}'.$$

Corollary 1.2.6. *For $v \in \mathcal{C}^1\left(\bar{\Omega}\right)$ a scalar function we have that*

$$\int_\Omega v(\mathbf{x})\nabla \cdot \mathbf{F}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} v(\sigma)\mathbf{F}(\sigma) \cdot \mathbf{n}(\sigma)\,\mathrm{d}\sigma - \int_\Omega \nabla v(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Corollary 1.2.7 (Green's First Identity)**.** *For $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1\left(\bar{\Omega}\right)$ a scalar function we have that*

$$\int_\Omega v(\mathbf{x})\Delta u(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} v(\sigma)\partial_{\mathbf{n}} u(\sigma)\,\mathrm{d}\sigma - \int_\Omega \nabla v(\mathbf{x}) \cdot \nabla u(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

where

- $\nabla \cdot \nabla u = \Delta u$, *and*
- $\nabla u \cdot \mathbf{n} = \partial_{\mathbf{n}} u$.

Corollary 1.2.8 (Green's Second Identity)**.** *For $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1\left(\bar{\Omega}\right)$ a scalar function we have that*

$$\int_\Omega v(\mathbf{x})\Delta u(\mathbf{x}) - u(\mathbf{x})\Delta v(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Omega v(\sigma)\partial_{\mathbf{n}} u(\sigma) - u(\sigma)\partial_{\mathbf{n}} v(\sigma)\,\mathrm{d}\sigma.$$

All of the above results also hold for Lipschitz domains.

1.2.1 Important Theorems

Theorem 1.2.9 (Cauchy-Lipschitz Existence and Uniqueness). *Fix $t_0 \in \mathbb{R}, y_0 \in \mathbb{R}^d, a, b > 0$ and let*

$$R = \{(t, y) : t_0 \leq t \leq t_0 + a, |y - y_0| \leq b\}.$$

Let f be continuous on R with $M = \max_{(t,y)\in R}(f(t, y))$, and uniformly Lipschitz continuous with respect to y. Then the differential equation

$$y'(t) = f(t, y(t))$$

with $y(t_0) = t_0$, has a unique solution $y(t)$ defined on $[t_0, t_0 + T]$ for $T = \min\left\{a, \frac{b}{M}\right\}$.

Definition 1.2.10. *For $\Omega \subset \mathbb{R}^d$ and $\mathbf{F} \in C^1(\Omega)$, the Differential of $\mathbf{F}$ at a point $\mathbf{x} \in \Omega$ is denoted $D\mathbf{F}(\mathbf{x})$ and is the $d \times d$ Jacobian matrix with entries*

$$[D\mathbf{F}]_{ij} = \partial_{x_j} F_i$$

for $i, j = 1, \ldots, d$.

Theorem 1.2.11 (Inverse Function Theorem). *Suppose $\mathbf{F} \in \mathcal{C}^1(\Omega)$, $D\mathbf{F}(\mathbf{p})$ is invertible at $\mathbf{p} \in \Omega$, and $\mathbf{q} = \mathbf{F}(\mathbf{p})$. Then, there exists open sets $U, V \subset \mathbb{R}^d$ with $\mathbf{p} \in U, \mathbf{q} \in V$ such that $\mathbf{F}$ is injective on U and $\mathbf{F}(U) = V$. More specifically, the inverse of $\mathbf{F}$ given by $\mathbf{G} : V \to U$ is such that $\mathbf{G} \in \mathcal{C}^1(V)$.*

Theorem 1.2.12 (Implicit Function Theorem). *For $\Omega \subset \mathbb{R}^{n+m}$, suppose $\mathbf{F} \in C^1(\Omega)$ maps into $\mathbb{R}^n$ so that $\mathbf{F}(\mathbf{p}, \mathbf{q}) = \mathbf{0}$ for some $(\mathbf{p}, \mathbf{q}) \in \Omega$. Assume that $D_{\mathbf{y}}\mathbf{F}(\mathbf{p}, \mathbf{q})$ is invertible. Then there exists open sets $V \subset \mathbb{R}^{n+m}$ and $U \subset \mathbb{R}^n$ with $(\mathbf{p}, \mathbf{q}) \in V$ and $\mathbf{p} \in U$ such that for every $\mathbf{x} \in U$ there is a unique $\mathbf{y}$ such that $(\mathbf{x}, \mathbf{y}) \in V$ and*

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}.$$

Letting $\mathbf{G}(\mathbf{x}) := \mathbf{y}$, we have that $\mathbf{G} \in \mathcal{C}^1(U)$ and

$$\mathbf{F}(\mathbf{x}, \mathbf{G}(\mathbf{x})) = 0$$

for $\mathbf{x} \in U$. Moreover,

$$D\mathbf{G}(\mathbf{x}) = -(D_{\mathbf{y}}\mathbf{F}(\mathbf{x}, \mathbf{G}(\mathbf{x})))^{-1} D_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{G}(\mathbf{x}))$$

for $\mathbf{x} \in U$.

1.3 Fourier Series

Definition 1.3.1. *The Fourier series of a periodic function $f : \mathbb{R} \to \mathbb{R}$ of period $2T > 0$ is*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{\pi k x}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k x}{T}\right)$$

where

$$a_k = \frac{1}{T}\int_{-T}^{T} f(x) \cos\left(\frac{\pi k x}{T}\right) \mathrm{d}x,$$

and

$$b_k = \frac{1}{T}\int_{-T}^{T} f(x) \sin\left(\frac{\pi k x}{T}\right) \mathrm{d}x.$$

For f an odd function, it follows that $a_k = 0$ for all $k \geq 0$ and we are left with the sine Fourier series

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k x}{T}\right)$$

with coefficients taking the simplified form

$$b_k = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{\pi k x}{T}\right) \mathrm{d}x.$$

For f an even function, it follows that $b_k = 0$ for all $k \geq 1$ and we are left with the cosine Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{\pi k x}{T}\right)$$

with coefficients taking the simplified form

$$a_k = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{\pi k x}{T}\right) \mathrm{d}x.$$

Proposition 1.3.2. *Let $f \in \mathcal{C}^1(\mathbb{R})$ be $2T$-periodic with Fourier coefficients a_k and b_k. Then f' has Fourier coefficients a'_k and b'_k given by*

$$a'_k = \frac{\pi k}{T} b_k,$$

and

$$b'_k = -\frac{\pi k}{T} a_k.$$

Theorem 1.3.3. *Assume that f is a square-integrable function defined on $(-T, T)$, that is*

$$\int_{-T}^{T} |f(x)|^2 \, \mathrm{d}x < \infty.$$

Then

$$\lim_{N \to \infty} \int_{-T}^{T} |S_N(x) - f(x)|^2 dx = 0,$$

where

$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos\left(\frac{\pi k x}{T}\right) + \sum_{k=1}^{N} b_k \sin\left(\frac{\pi k x}{T}\right).$$

Moreover,

$$\frac{1}{T} \int_{-T}^{T} |f(x)|^2 \, \mathrm{d}x = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left(|a_k|^2 + |b_k|^2\right)$$

and

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} b_k = 0.$$

Theorem 1.3.4. *If $f \in \mathcal{C}^1\left([-T, T]\right)$, then*

$$\sum_{k=1}^{\infty} |a_k|,$$

and

$$\sum_{k=1}^{\infty} |b_k|$$

are convergent. In particular, the Fourier series of f is uniformly convergent in $\mathbb{R}$ to f.

2 First Order Equations

2.1 Transport Equations

2.1.1 Pure Transport

The pure transport equation is defined as

$$\partial_t \rho + a\partial_x \rho = 0 \tag{2.1.1}$$

where $\rho(t,x)$, and $a > 0$. Moreover, the initial conditions are specified by

$$\rho(0,x) = g(x)$$

for some assignment function g. Let

$$\mathbf{a} = \begin{pmatrix} a \\ 1 \end{pmatrix}$$

so that (2.1.1) becomes $\nabla\rho \cdot \mathbf{a} = 0$. As the gradient of the function is perpendicular to its level lines, it follows that the level lines of ρ are the straight lines parallel to $\mathbf{a}$, namely,

$$x = at + x_0 \tag{2.1.2}$$

for arbitrary $x_0 \in \mathbb{R}^d$. The lines represented by (2.1.2) are known as the characteristics. Indeed one has that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t, at + x_0) = (\partial_t \rho + a\partial_x \rho)\,(t + at + x_0) = 0.$$

Hence, for a particular initial condition $\rho(0,x_0) = g(x_0)$, as ρ is constant along the characteristic lines we must have

$$\rho(t, at + x_0) = \rho(0, x_0) = g(x_0).$$

Therefore,

$$\rho(t,x) = g(x - at).$$

2.1.2 Distributed Source

Consider the initial value problem

$$\begin{cases} \partial_t \rho + a\partial_x \rho = f(t,x) & t > 0,\, x \in \mathbb{R} \\ \rho(0,x) = g(x) & x \in \mathbb{R}. \end{cases} \tag{2.1.3}$$

Exercise 2.1.1. *Show that the solution to* (2.1.3) *is given by*

$$\rho(t,x) = g(x - at) + \int_0^t f(s, x - a(t-s))\,\mathrm{d}s.$$

2.1.3 Damped Travelling Waves

Consider the initial value problem

$$\begin{cases} \partial_t \rho + a\partial_x \rho = -\delta\rho & t > 0, x \in \mathbb{R} \\ \rho(0,x) = g(x) & x \in \mathbb{R}. \end{cases} \tag{2.1.4}$$

Exercise 2.1.2. *Show that the solution to* (2.1.4) *is given by*

$$\rho(t,x) = \exp(-\delta t) g(x - at).$$

2.2 Method of Characteristics

We can generalise the approach taken to solve the transport equations, to what is known as the method of characteristics. Consider the quasi-linear equation

$$a(x, y, u)\partial_x u + b(x, y, u)\partial_y u = c(x, y, u) \tag{2.2.1}$$

where $x, y \in \Omega \subset \mathbb{R}^2$ and $a, b, c : \mathbb{R}^3 \to \mathbb{R}$ are continuously differentiable. The method of characteristics utilises the intuition to solve (2.2.1). Let (x_0, y_0, z_0) be a point on the graph of u. Then the tangent plane to the graph u at (x_0, y_0, z_0) has the equation

$$\partial_x u(x_0, y_0)(x - x_0) + \partial_y u(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Then the vector normal to the place

$$\mathbf{n}_0 = \begin{pmatrix} \partial_x u(x_0, y_0) \\ \partial_y u(x_0, y_0) \\ -1 \end{pmatrix}$$

is is also normal to the vector

$$\mathbf{v}_0 = \begin{pmatrix} a(x_0, y_0, z_0) \\ b(x_0, y_0, z_0) \\ c(x_0, y_0, z_0) \end{pmatrix}$$

by the fact that u satisfies (2.2.1). Therefore, $\mathbf{v}_0$ is tangent to the graph of u at (x_0, y_0, z_0) and thus the graph of any solution is tangent to the vector field

$$\mathbf{v} = (a(x, y, z), b(x, y, z), c(x, y, z)).$$

Re-parameterising the curve of u as $(x(s), y(s), z(s))$ where $z(s) = u(x(s), y(s))$, it follows that the tangent vector of this curve must $\mathbf{v}$ so that

$$\begin{cases} \frac{dx}{ds} = a(x, y, z), \\ \frac{dy}{ds} = b(x, y, z), \\ \frac{dz}{ds} = c(x, y, z). \end{cases}$$

The curves $(x(s), y(s), z(s))$ are the called the characteristics of (2.2.1).

Exercise 2.2.1. *Verify that $z(s)$ gives the solution, u, to* (2.2.1) *along a characteristic. That is,*

$$\frac{\mathrm{d}z}{\mathrm{d}s} = c(x(s), y(s), z(s)).$$

Thus along the characteristics (2.2.1) *reduces to an ordinary differential equation.*

2.2.1 Cauchy Problem for First Order Quasi-linear Equations

Let $I \subset \mathbb{R}$ be an interval containing 0, and let $\gamma : I \to \mathbb{R}^2$ be a smooth curve parameterised by

$$\gamma(\tau) = (\gamma_1(\tau), \gamma_2(\tau)) \in \mathbb{R}^2,$$

where γ_1, γ_2 are continuously differentiable in I. Consider the Cauchy problem

$$\begin{cases} a(x, y, u)\partial_x u + b(x, y, u)\partial_y u = c(x, y, u) & (x, y) \in \mathbb{R}^2 \\ u(\gamma_1(\tau), \gamma_2(\tau)) = g(\tau) & \tau \in I. \end{cases} \tag{2.2.2}$$

We can solve (2.2.2) by applying the method of characteristics in the following way.

1. Using Theorem 1.2.9 there is a unique solution

$$\begin{cases} x = X(s,\tau), \\ y = Y(s,\tau), \\ z = Z(s,\tau), \end{cases}$$

   for the system of characteristics

$$\begin{cases} \frac{dx}{ds} = a(x,y,z) & x(0) = \gamma_1(\tau) \\ \frac{dy}{ds} = b(x,y,z) & y(0) = \gamma_2(\tau) \\ \frac{dz}{ds} = c(x,y,z) & z(0) = g(\tau) \end{cases}$$

   in a neighbourhood of $s = 0$ and for $\tau \in I$.

2. For $\tau_0 \in I$ check that

$$\begin{vmatrix} \partial_s X(0,\tau_0) & \partial_s Y(0,\tau_0) \\ \gamma_1'(\tau_0) & \gamma_2'(\tau_0) \end{vmatrix} \neq 0$$

   so that the above (X, Y, Z) defines a function u.

Theorem 2.2.2. *Let $a, b, c \in \mathcal{C}^1$ in a neighbourhood of $(x_0, y_0, u(x_0, y_0))$ and assume that $\gamma_1, \gamma_2, g \in \mathcal{C}^1(I)$. If*

$$\begin{vmatrix} \partial_s X(0,\tau_0) & \partial_s Y(0,\tau_0) \\ \gamma_1'(\tau_0) & \gamma_2'(\tau_0) \end{vmatrix} \neq 0$$

then in a neighbourhood of (x_0, y_0) there exists a unique solution $\mathcal{C}^1$-solution $u = u(x, y)$ of the Cauchy problem

$$\begin{cases} a(x,y,u)\partial_x u + b(x,y,u)\partial_y u = c(x,y,u) & (x,y) \in \mathbb{R}^2 \\ u(\gamma_1(\tau), \gamma_2(\tau)) = g(\tau) & \tau \in I. \end{cases}$$

Moreover, $u(x, y)$ is defined by the parametric equations

$$\begin{cases} x = X(s,\tau), \\ y = Y(s,\tau), \\ z = Z(s,\tau). \end{cases}$$

2.2.2 Cauchy Problem for First Order Linear Equations

Consider the linear equation

$$\begin{cases} a(x,y)\partial_x u + b(x,y)\partial_y u = c(x,y) & (x,y) \in \mathbb{R}^2 \\ u(\gamma_1(\tau), \gamma_2(\tau)) = g(\tau) & \tau \in I. \end{cases}$$

The system of characteristics is given by

$$\begin{cases} \frac{dx}{ds} = a(x,y) & x(0) = \gamma_1(\tau) \\ \frac{dy}{ds} = b(x,y) & y(0) = \gamma_2(\tau) \\ \frac{dz}{ds} = c(x,y) & z(0) = g(\tau) \end{cases}$$

from which we observe that z is decoupled. For the first two we can determine a solution

$$\begin{cases} x = X(s,\tau), \\ y = Y(s,\tau) \end{cases}$$

and thus

$$Z = Z(s,\tau) = g(\tau) + \int_0^s C(X(\sigma,\tau), Y(\sigma,\tau))\,\mathrm{d}\sigma.$$

After checking that the inversion of the system of equations is valid we can conclude that

$$u(X(s,\tau),Y(s,\tau)) = g(\tau) + \int_0^s c(X(\sigma,\tau),Y(\sigma,\tau))d\sigma.$$

Exercise 2.2.3. *Find the unique to solution to the Cauchy problem*

$$\begin{cases} -y\partial_x u + x\partial_y u = 4xy & (x,y)\in\mathbb{R}^2 \\ u(x,0) = g(x) & x > 0. \end{cases}$$

2.3 Scalar Conservation Laws

The non-linear generalization of the pure transport equation

$$\partial_t \rho + \partial_x q(\rho) = 0$$

is known as the scalar conservation law. If ρ is differentiable we can investigate this law with the Cauchy problem

$$\begin{cases} \partial_t \rho + q'(\rho)\partial_x \rho = 0 & x\in\mathbb{R}, t>0 \\ \rho(0,x) = g(x) & x\in\mathbb{R}. \end{cases} \tag{2.3.1}$$

Consequently, we can write down the system of characteristics

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}s} = q'(z) & x(0)=\tau, \\ \frac{\mathrm{d}t}{\mathrm{d}s} = 1 & t(0)=0, \\ \frac{\mathrm{d}z}{\mathrm{d}s} = 0 & z(0) = g(\tau). \end{cases}$$

Solving these we see that the characteristic lines are given by

$$x = \tau + q'(g(\tau))t.$$

Issues arise at points where the characteristics for different values τ intersect, for instance, they introduce discontinuities into the solution.

2.3.1 Existence of Classical Solutions

Definition 2.3.1. *For $T>0$, a classical solution to (2.3.1) is a function $\mathcal{C}^1([0,T)\times\mathbb{R})$ that satisfies (2.3.1) for every $(t,x)\in(0,T)\times\mathbb{R}$. In particular, if $T=\infty$ then the solution is a global-in-time classical solution.*

Theorem 2.3.2. *Let $q\in\mathcal{C}^2(\mathbb{R})$ and $g\in\mathcal{C}^1(\mathbb{R})$, and assume that there exists $M>0$ so that*

- $\sup_{r\in\mathbb{R}}|q''(r)|\le M$*, and*
- $\sup_{r\in\mathbb{R}}|g'(r)|\le M$.

Then there exists $T=T(M)>0$ such that there is a unique classical solution, ρ, to (2.3.1) defined on $[0,T]\times\mathbb{R}$. Moreover, if

$$q''(r)g'(s)\ge 0$$

for all $r,s\in\mathbb{R}$ then the solution is global in time.

Example 2.3.3. *Burger's equation is the Cauchy problem,*

$$\begin{cases} \partial_t \rho + \rho\partial_x \rho = 0 & x\in\mathbb{R},\ t>0 \\ \rho(0,x) = g(x) & x\in\mathbb{R}. \end{cases} \tag{2.3.2}$$

If g is increasing the $g'(s) \geq 0$ for all $s \in \mathbb{R}$ and as $q''(r) = 1 \geq 0$, the condition of Theorem 2.3.2 is satisfied. Therefore, there exists a global solution to (2.3.2)*. Now suppose that g is not increasing, such that there exists $\tau_1 < \tau_2$ with the property that $g(\tau_1) > g(\tau_2)$. Recall, that the characteristics at τ_1 and τ_2 are given by*

$$x = \tau_1 + g(\tau_1)t$$

and

$$x = \tau_2 + g(\tau_2)t$$

respectively. These intersect at

$$T := \frac{\tau_2 - \tau_1}{g(\tau_1) - g(\tau_2)} > 0.$$

Therefore, there is a discontinuity in the solution, which implies that solution to (2.3.2) *cannot be a classical solution global in time. However, it turns out that the solution remains classical for $t < T$, and it is only at the point T where the solution ceases to be a classical solution. Such a time and spatial location, (x_s, t_s), where the solution ceases to be a $\mathcal{C}^1$ function, is known as a shock.*

2.3.2 Weak Solutions

Example 2.3.3 illustrates the limitations of the method of characteristics for determining solutions for all initial conditions. To circumvent these issues we relax the regularity conditions we impose on solutions. To explore this idea consider the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x q(\rho) = 0 & x \in \mathbb{R}, t > 0 \\ \rho(0, x) = g(x) & x \in \mathbb{R}. \end{cases} \tag{2.3.3}$$

Let v be a smooth function on $[0, \infty) \times \mathbb{R}$, called a test function. One can show that for any smooth solution ρ satisfying (2.3.3) we have

$$\int_0^\infty \int_{\mathbb{R}} (\partial_t \rho + \partial_x q(\rho)) v dx dt + \int_{\mathbb{R}} (g(x) - \rho(0, x)) v(0, x) dx = 0. \tag{2.3.4}$$

Exercise 2.3.4. *Verify that if a bounded function on $[0, \infty) \times \mathbb{R}$, ρ, satisfies* (2.3.4) *if and only if it is a solution to* (2.3.3)*.*

Definition 2.3.5. *A function ρ, bounded on $[0, \infty) \times \mathbb{R}$ is a weak solution to* (2.3.3) *if* (2.3.4) *holds for every test function v on $[0, \infty) \times \mathbb{R}$ with compact support.*

Remark 2.3.6. *Note that the only condition imposed on ρ is that it is bounded, that is, it can be discontinuous.*

2.3.3 The Rankine-Hugoniot Condition

For an open set V, contained in the half-plane $t > 0$, consider the domains V_+ and V_- partitioned by a smooth shock curve $\Gamma : x = \sigma(t)$. Suppose ρ is a classical solution on both sides of Γ, with continuous derivatives. Take a test function v supported on a compact set $K \subset V$, with $K \cap \Gamma \neq \emptyset$. One can then show that the outward normal, $\mathbf{n}$, to ∂V_+ is

$$\mathbf{n} = \frac{1}{\sqrt{1 + |\sigma'(t)|^2}} \begin{pmatrix} -1 \\ \sigma'(t) \end{pmatrix}$$

and

$$\sigma' = \frac{q(\rho_+(t, \sigma)) - q(\rho_-(t, \sigma))}{\rho_+(t, \sigma) - \rho_-(t, \sigma)}$$

where $\rho_\pm$ is the value of ρ on V from the $V_\pm$ side. This condition on Γ is known as the Rankine-Hugoniot Condition. One can solve this condition to find the equation of a shock for a particular problem.

Example 2.3.7. *Consider Burgers equation*

$$\begin{cases} \partial_t \rho + \rho \partial_x \rho = 0 & x \in \mathbb{R}, t > 0 \\ \rho(0,x) = g(x) & x \in \mathbb{R}. \end{cases}$$

with $g(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0. \end{cases}$ *It can be shown that the characteristics intersect along straight lines, forming a shock. As* $\rho_+ = 0$ *and* $\rho_- = 1$*,we can solve the Rankine-Hugoniot condition to get the shock*

$$x = \sigma(t) = \frac{t}{2}.$$

2.3.4 The Entropy Condition

Solutions satisfying the Rankine-Hugoniot condition may not be unique.

Definition 2.3.8. *Let* ρ *be a weak solution to* (2.3.3)*, with a discontinuous shock* $\Gamma : x = \sigma(t)$*. The shock is entropic if*

$$q'(\rho_+(t, \sigma(t))) < \sigma'(t) < q'(\rho_-(t, \sigma(t))),$$

for every t *for which the shock is defined.*

Remark 2.3.9. *A shock is entropic if the slope of the shock curve is less than the slope of the left characteristics and greater than the slope of the right characteristics. Consequently, the characteristics hit the shock curve forward in time, and it is not possible to go back in time along a characteristic that hits a shock line.*

Theorem 2.3.10. *If* $q \in \mathcal{C}^2(\mathbb{R})$ *is convex (or concave) and* g *is bounded, then there exists a unique entropic solution to the problem*

$$\begin{cases} \partial_t \rho + \partial_x q(\rho) = 0 & x \in \mathbb{R}, t > 0 \\ \rho(0,x) = g(x) & x \in \mathbb{R}. \end{cases}$$

Example 2.3.11. *Consider the car traffic on roads.*

- ρ *is the car density.*
- v *is their average speed.*
- q *is their flux.*

These three quantities are linked via $q = v\rho$*. We make the following assumptions.*

1. *Average speed is a function of* ρ*, with* $v'(\rho) \leq 0$*.*
2. *There is a maximum velocity* $v_m > 0$*.*
3. *Traffic stops at a maximum density* $\rho > 0$*.*

Under these assumptions we suppose that $v(\rho) = v_m \left(1 - \frac{\rho}{\rho_m}\right)$ *which means that* $q(\rho) = v_m \rho \left(1 - \frac{\rho}{\rho_m}\right)$*. Consequently, one can show that* q *is concave. The conservation laws for this scenario may be summarised by* *the problem*

$$\begin{cases} \partial_t \rho + v_m \left(1 - \frac{2\rho}{\rho_m}\right) \partial_x \rho = 0 & x \in \mathbb{R}, t > 0 \\ \rho(0, x) = g(x) & x \in \mathbb{R} \end{cases}$$

which has the corresponding characteristics

$$x = \tau + q'(g(\tau))t$$

for $\tau \in \mathbb{R}$. We can model different scenarios by specifying $g(x)$. For example,

$$g(x) = \begin{cases} \rho_m & x \leq 0 \\ 0 & x > 0 \end{cases}$$

could represent traffic waiting at a red light, whilst the road ahead is clear. On the other hand,

$$g(x) = \begin{cases} \frac{\rho_m}{8} & x \leq 0 \\ \rho_m & x > 0 \end{cases}$$

could represent a traffic jam. We will first consider the case of traffic waiting at a red light. The characteristics in this setting may be written as

$$x = \begin{cases} \tau - v_m t & \tau \leq 0 \\ \tau + v_m t & \tau > 0. \end{cases}$$

These characteristics do not cover the region $S = \{(x,t) : |x| < v_m t\}$. We fill in this vacant region using a rarefaction wave. That is, we look for a solution of the form $\rho(t,x) = h\left(\frac{x}{t}\right)$. Using this form for ρ we conclude that the unique entropy solution is given by

$$\rho(t,x) = \begin{cases} \rho_m & x \leq -v_m t \\ \frac{\rho_m}{2}\left(1 - \frac{x}{v_m t}\right) & -v_m t < x < v_m t \\ 0 & x \geq v_m t. \end{cases}$$

Now we consider the case of a traffic jam. Similar to the previous case we see that the characteristics are

$$x = \begin{cases} \tau + \frac{3}{4} v_m t & \tau \leq 0 \\ \tau - v_m t & \tau > 0. \end{cases}$$

These characteristics intersect and so form a shock. From the Rankine-Hugoniot condition we obtain

$$x = \sigma(t) = -\frac{v_m}{t}$$

and so the unique entropy solution is given by

$$\rho(t,x) = \begin{cases} \frac{\rho_m}{8} & x < -\frac{v_m}{8} t \\ \rho_m & x > -\frac{v_m}{8} t. \end{cases}$$

2.4 The Continuity Equations

The multi-dimensional generalization of the pure transport equation for a smooth domain $\Omega \subset \mathbb{R}^d$ and smooth vector field $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) : \mathbb{R} \times \Omega \to \mathbb{R}^d$ is

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 & t > 0, \mathbf{x} \in \Omega \\ \rho(0, \mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \Omega \end{cases} \tag{2.4.1}$$

This is known as the continuity equation. The form of the continuity equation can be simplified depending on the physical situation we are modelling.

2.4.1 Particle Trajectories

For an initial configuration of particles, $\mathbf{a} \in \Omega$, their trajectories can be modelled by

$$\mathbf{X}(t, \mathbf{a}) = (X_1, \dots, X_d)(t, \mathbf{a}) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d.$$

The map $\mathbf{a} \mapsto \mathbf{X}(t, a)$ is known as the flow map. Under suitable conditions we have the back-to-labels map, $\mathbf{A}$, which satisfies

$$\begin{cases} \mathbf{A}(t, \mathbf{X}(t, \mathbf{a})) = \mathbf{a}, \\ \mathbf{X}(t, \mathbf{A}(t, \mathbf{x})) = \mathbf{x} \end{cases}$$

for all $\mathbf{a}, \mathbf{x} \in \mathbb{R}^d$. We can think of the particles as moving along a velocity field, $\mathbf{u}$, according to

$$\partial_t \mathbf{X}(t, \mathbf{a}) = \mathbf{u}(t, \mathbf{X}(t, a))$$

and

$$X(0, \mathbf{a}) = \mathbf{a}.$$

For a function f, let the convective derivative along the flow be

$$\begin{aligned} \partial_t(f(t, \mathbf{X}(t, \mathbf{a}))) &= (D_t f)(t, \mathbf{X}(t, \mathbf{a})) \\ &= \partial_t f + \mathbf{u} \cdot \nabla f, \end{aligned}$$

where the second equality comes from the chain rule, which applies if we assume all functions are C^1. Using this we observe that (2.4.1) is equivalent to

$$\begin{cases} D_t \rho = -\rho \nabla \cdot \mathbf{u} & t > 0, \mathbf{x} \in \Omega \\ \rho(0, \mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \Omega. \end{cases} \tag{2.4.2}$$

Let

$$J(t, \mathbf{a}) = \det\left(\nabla_{\mathbf{a}} \mathbf{X}\right)(t, \mathbf{a}).$$

Lemma 2.4.1. *Suppose we have $\mathbf{u}$ in $\mathcal{C}^1$ defining $\mathbf{X}$ and J be as above. Then*

$$\partial_t J(t, \mathbf{a}) = J(t, \mathbf{a})(\nabla \cdot \mathbf{u}(t, \mathbf{X}(t, \mathbf{a})))$$

pointwise in $(t, \mathbf{a})$.

Lemma 2.4.2. *Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^1$ function, and assume the $\mathbf{u}$ defining $\mathbf{X}(t, \cdot)$ is also in $\mathcal{C}^1$. Then we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V(t)} f(t, \mathbf{x}) \, \mathrm{d}\mathbf{x}\right) = \int_{V(t)} (\partial_t f + \nabla \cdot (f \mathbf{u}))(t, \mathbf{x}) \, \mathrm{d}\mathbf{x}$$

for every $t > 0$ and fluid element V where

$$V(t) = \mathbf{X}(t, V) = \{\mathbf{X}(t, \mathbf{a}) : \mathbf{a} \in V\}.$$

Applying Lemma 2.4.2 to the total mass

$$m(t, V) = \int_V \rho(t, \mathbf{x}) \, \mathrm{d}\mathbf{x}$$

we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t, V(t)) = 0.$$

Thus, mass is neither created or destroyed. In this scenario, ρ should be thought of as the density of concentration.

2.4.2 Flow Compressibility

Definition 2.4.3. *A velocity field* $\mathbf{u}$ *is incompressible if the flow map* $\mathbf{X}(t,\cdot)$ *is such that*

$$|V| = |V(t)|$$

for all $V \subset \Omega$ *and* $t \geq 0$*, where* $|V|$ *is the Lebesgue measure of the set* V*.*

Proposition 2.4.4. *The velocity field* $\mathbf{u}$ *is incompressible if and only if* $\nabla \cdot \mathbf{u} = 0$ *for every* $\mathbf{x} \in \Omega$ *and* $t \geq 0$*. In such a case* $\mathbf{u}$ *is said to be divergence-free.*

Corollary 2.4.5. *Suppose we have a divergence free vector field* $\mathbf{u}$ *in* $\mathcal{C}^1$ *defining* $\mathbf{X}$ *and* J*, then*

$$J(t, \mathbf{a}) = 1$$

for all $\mathbf{a} \in \Omega$ *and* $t > 0$*.*

If we assume the fluid is incompressible then we can solve (2.4.2) to yield the solution

$$\rho(t, \mathbf{x}) = g(\mathbf{A}(t, \mathbf{x})).$$

On the other hand, if $\nabla \cdot \mathbf{u} \neq 0$ then

$$\rho(t, \mathbf{X}(t, \mathbf{a})) = g(\mathbf{a}) \exp\left(-\int_0^t \nabla \cdot \mathbf{u}(s, \mathbf{X}(s, \mathbf{a}))\,\mathrm{d}s\right)$$

and

$$J(t, \mathbf{a}) = \exp\left(\int_0^t t, \mathbf{X}(t, \mathbf{a})\,\mathrm{d}s\right).$$

Therefore,

$$\rho(t, \mathbf{x}) = \frac{g(\mathbf{A}(t, \mathbf{x}))}{J(t, \mathbf{A}(t, \mathbf{x}))}.$$

2.5 Solution to Exercises

Exercise 2.1.1

Solution. For the characteristic lines

$$x = at + x_0,$$

where $x_0 \in \mathbb{R}$ we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t, at + x_0) = f(t, at + x_0).$$

Therefore,

$$\begin{aligned}\rho(t, at + x_0) &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\rho(t, at + x_0)\,\mathrm{d}s + c \\ &= \int_0^t f(s, as + x_0)\,\mathrm{d}s + c.\end{aligned}$$

With the initial condition that $\rho(0, x_0) = g(x_0)$ it follows that $c = g(x_0)$ and so

$$\rho(t, at + x_0) = g(x_0) + \int_0^t f(s, as + x_0)\,\mathrm{d}s.$$

Letting $x = at + x_0$ we deduce that

$$\rho(t,x) = g(x - at) + \int_0^t f(s, x - a(t-s))\,\mathrm{d}s.$$

□

Exercise 2.1.2

Solution. For the characteristic lines

$$x = at + x_0,$$

where $x_0 \in \mathbb{R}$ we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t, at + x_0) = -\delta\rho(t, at + x_0).$$

This of the form $y' = -\delta y$, which can be solved through separation of variables to yield

$$e^{\delta t}\rho(t, at + x_0) = g(x_0)$$

where we have used the initial condition $\rho(0, x_0) = g(x_0)$ to fit the constant of integration. Letting $x = at + x_0$ it follows that

$$\rho(t,x) = e^{-\delta t} g(x - at).$$

□

Exercise 2.2.3

Solution. In this case $I = \{(x,y) : x > 0, y = 0\}$ with $\gamma_1(\tau) = \tau$ and $\gamma_2(\tau) = 0$ for $\tau \in (0, \infty)$. Therefore the characteristic system is

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}s} = -y & x(0) = \tau \\ \frac{\mathrm{d}y}{\mathrm{d}s} = x & y(0) = 0 \\ \frac{\mathrm{d}z}{\mathrm{d}s} = 4xy & z(0) = g(\tau). \end{cases}$$

We can deduce that

$$x = X(s,\tau) = \tau\cos(s) \text{ and } y = Y(s,\tau) = \tau\sin(s).$$

from which we get that

$$u(X(s,\tau), Y(s,\tau)) = g(\tau) + 2\tau^2\sin^2(s).$$

Using the substitution $s = \arctan\left(\frac{y}{x}\right)$ and $\tau = \sqrt{x^2 + y^2}$ we conclude that

$$u(x,y) = g\left(\sqrt{x^2 + y^2}\right) + 2y^2.$$

□

Exercise 2.2.1

Solution. Proceeding directly we observe that

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}s} &= \partial_x u(x(s), y(s))\frac{\mathrm{d}x}{\mathrm{d}s} + \partial_y u(x(s), y(s))\frac{\mathrm{d}y}{\mathrm{d}s} \\ &= a(x(s), y(s), z(s))\partial_x u(x(s), y(s)) + b(x(s), y(s), z(s))\partial_y u(x(s), y(s)) \\ &= c(x(s), y(s), z(s)). \end{aligned}$$

□

Exercise 2.3.4

Solution. Suppose that ρ satisfies (2.3.3). Let v be a smooth function on $[0,\infty) \times \mathbb{R}$ with compact support. From (2.3.3) it follows that

$$\int_0^\infty \int_{\mathbb{R}} (\partial_t \rho + \partial_x (q(\rho)))v \,\mathrm{d}x \,\mathrm{d}t = 0.$$

Integrating by parts with respect to both t and x, and using the fact that v has compact support we get that

$$\int_0^\infty \int_{\mathbb{R}} \rho \partial_t v + q(\rho)\partial_x v \,\mathrm{d}x \,\mathrm{d}t = -\int_{\mathbb{R}} g(x)v(0,x)\,\mathrm{d}x.$$

Conversely, suppose that ρ satisfies (2.3.4). Integrating by parts in the reverse order yields

$$\int_0^\infty \int_{\mathbb{R}} (\partial_t \rho + \partial_x (q(\rho)))v \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} (g(x) - \rho(0,x))v(0,x)\,\mathrm{d}x = 0.$$

Choosing v to vanish at $t = 0$, the second equation is zero, and so we deduce that

$$\int_0^\infty \int_{\mathbb{R}} (\partial_t \rho + \partial_x (q(\rho)))v \,\mathrm{d}x \,\mathrm{d}t = 0$$

for all such v which implies that ρ satisfies (2.3.3). $\square$

3 The Diffusion Equation

3.1 The Diffusion Equations

The diffusion equation is a linear second-order partial differential equation

$$\partial_t u(t, \mathbf{x}) - \kappa \Delta u(t, \mathbf{x}) = f(t, \mathbf{x}) \tag{3.1.1}$$

where

- $\mathbf{x}(x_1, \dots, x_d) \in \mathbb{R}^d$ is the space variable,
- t is the time variable,
- κ is a positive diffusion coefficient, and
- $\Delta = \sum_{i=1}^n \partial_{x_i x_i}$ is the Laplace operator.

3.1.1 Heat Conduction

We can apply (3.1.1) to explore the heat conduction of a solid body. As heat is a form of energy, the law of conservation applies. We operate under the following assumptions.

1. The body has volume V.
2. The body is of constant mass density ρ.
3. The rate per unit of supplied heat by an external source is r.
4. The thermal energy per unit mass is $e = e(t, \mathbf{x})$, is linear function of temperature. More specifically $e = c_0 u$ where c_0 is the specific heat of the material.
5. The heat flux, that is the magnitude and direction of heat flow, is $\mathbf{q}$ and satisfies Fourier's Law which says that $\mathbf{q} = -\kappa_F \nabla u$ for $\kappa_F > 0$.
6. $\mathrm{d}\sigma$ is an area element contained in ∂V with outer unit normal $\mathbf{n}$, meaning $\mathbf{q} \cdot \mathbf{n} \mathrm{d}\sigma$ is the energy flow rate through $\mathrm{d}\sigma$.

The conservation of energy tells us that

$$\int_V \rho \partial_t e \, \mathrm{d}\mathbf{x} = -\int_V \nabla \cdot \mathbf{q} \, \mathrm{d}\mathbf{x} + \int_V \rho r \, \mathrm{d}\mathbf{x}$$

for all V and hence

$$\rho \partial_t e = -\nabla \cdot \mathbf{q} + \rho r.$$

Which under our assumptions simplifies to

$$\partial_t u = \frac{\kappa_F}{\rho c_0} \Delta u + \frac{r}{c_0}. \tag{3.1.2}$$

3.2 Problems on Bounded Domains

We will explore problems on bounded space and time domains in the context of heat conduction. Therefore, we will adopt the notation and assumptions set out previously. However, we will also consider the following additional assumptions.

1. $\Omega \subset \mathbb{R}^d$ is a bounded domain.
2. Time spans the interval $[0, T]$.
3. $u = u(t, \mathbf{x})$ satisfies (3.1.2).

We will let $Q_T = (0,T) \times \Omega$ and prescribe the initial condition

$$u(0, \mathbf{x}) = g(\mathbf{x})$$

for $\mathbf{x} \in \bar{\Omega}$. We will refer to

$$\partial Q_T = \left(\{t=0\} \times \bar{\Omega}\right) \cup ((0,T] \times \partial\Omega)$$

as the parabolic boundary of Q_T.

Condition	Informal Definition	Formal Definition
Dirichlet	Temperature is constant on the boundary.	$u(t,\sigma) = h(t,\sigma)$ for $\sigma \in \partial\Omega, t \in (0,T]$ and some function h.
Neumann	The heat flux through the boundary is known.	$\partial_{\mathbf{n}} u(t,\sigma) = h(t,\sigma)$ for $\sigma \in \partial\Omega, t \in (0,T]$ and some function h.
Robin	Surrounding temperature is constant the heat enters the boundary linearly on the temperature difference.	$\partial_{\mathbf{n}} u(t,\sigma) + \alpha u(t,\sigma) = h(t,\sigma)$ for $\sigma \in \partial\Omega, t \in (0,T]$ and h some function, and $\alpha = \frac{\gamma}{\kappa_F}$.

3.2.1 One-Dimensional Dirichlet Problem

Consider the initial boundary value problem with homogeneous Dirichlet conditions,

$$\begin{cases} \partial_t u - \kappa \partial_{xx} u = 0 & (t,x) \in (0,T) \times (0,L) \\ u(t,0) = u(t,L) = 0 & t \in (0,T) \\ u(0,x) = g(x) & x \in [0,L]. \end{cases} \tag{3.2.1}$$

Theorem 3.2.1. *Let $\kappa > 0$ be a constant, and $g \in \mathcal{C}^1([0,L])$ with $g(0) = g(L) = 0$. The function*

$$u(t,x) = \sum_{n=1}^{\infty} B_n \exp\left(-\kappa \frac{n^2\pi^2}{L^2}\right) \sin\left(\frac{\pi n x}{L}\right),$$

for

$$B_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{\pi n x}{L}\right)$$

is of class $\mathcal{C}([0,\infty) \times [0,L]) \cap \mathcal{C}^\infty((0,\infty) \times [0,L])$ and solves (3.2.1).

Proof. Consider a function of the form $u(t,x) = w(t)v(x)$. Then from the first equation in (3.2.1) we have that

$$\frac{1}{\kappa} \frac{w'(t)}{w(t)} = \frac{v''(x)}{v(x)}.$$

As each side is dependent on different variables, it must be the case that for a constant λ we have

$$v'' - \lambda v = 0$$

with $v(0) = v(L) = 0$ and

$$w' - \lambda \kappa w = 0.$$

We can now consider different cases.

1. For $\lambda = 0$, we have $v(x) = A + Bx$ for which the boundary conditions imply that $A = B = 0$.
2. For $\lambda = \mu^2 > 0$, we have $v(x) = A\exp(-\mu x) + B\exp(\mu x)$ for which the boundary conditions imply that $A = B = 0$.
3. For $\lambda = -\mu^2 < 0$, we have $v(x) = A\cos(\mu x) + B\sin(\mu x)$ for which the boundary conditionals imply that $A = 0$, $B \in \mathbb{R}$ and $\mu = \mu_n = \frac{\pi n}{L}$ for $n = 1, 2, \ldots$. Moreover, we have that $w_n(t) = B\exp(-\kappa \mu_n^2 t)$.

□

From above we have obtained the family of solutions

$$u_n(t,x) = \exp\left(-\frac{\kappa n^2\pi^2 t}{L^2}\right)\sin\left(\frac{\pi n x}{L}\right)$$

which we superimpose to define

$$u(t,x) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{\kappa n^2\pi^2 t}{L^2}\right)\sin\left(\frac{\pi n x}{L}\right).$$

Here the B_n are arbitrary, however, if we fix them as stated in the theorem then we see that the initial condition $u(0,x) = g(x)$ is satisfied. Due to the fact that the odd extension, $\tilde{g}$, of g on $[-L,L]$ is such that that $\tilde{g} \in C^1([-L,L])$ and that $\tilde{g}(-L) = \tilde{g}(L) = 0$, so that its Fourier coefficients exist and are given by

$$\tilde{B}_n = \frac{1}{L}\int_{-L}^{L} \tilde{g}(x)\sin\left(\frac{\pi n x}{L}\right) = \frac{2}{L}\int_{0}^{L} g(x)\sin\left(\frac{\pi n x}{L}\right) = B_n$$

for $n \geq 1$. Finally, one can check that the second equation of (3.2.1) is satisfied by differentiating term-wise.

3.2.2 One-Dimensional Neumann Problem

Consider the initial boundary value problem with homogeneous Neumann conditions,

$$\begin{cases} \partial_t u - \kappa \partial_{xx} u = 0 & (t,x) \in (0,T)\times(0,L) \\ \partial_x u(t,0) = \partial_x u(t,L) = 0 & t \in (0,T) \\ u(0,x) = g(x) & x \in [0,L]. \end{cases} \tag{3.2.2}$$

Theorem 3.2.2. *Let $\kappa > 0$ be a constant and $g \in \mathcal{C}^1([0,L])$ with $g'(0) = g'(L) = 0$. The function*

$$u(t,x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \exp\left(-\kappa\frac{n^2\pi^2}{L^2}t\right)\cos\left(\frac{\pi n x}{L}\right)$$

is of class $\mathcal{C}([0,\infty)\times[0,L]) \cap \mathcal{C}^\infty((0,\infty)\times[0,L])$ and solves (3.2.1).

Proof. We proceed in the same in the proof of Theorem 3.2.1. That is,

$$v'' - \lambda v = 0$$

with $v'(0) = v'(L) = 0$ and

$$w' - \lambda\kappa w = 0.$$

We can now consider different cases.

1. For $\lambda = 0$, we have $v(x) = A + Bx$ for which the boundary conditions imply that $B = 0$ and $v(x) = A$.
2. For $\lambda = \mu^2 > 0$, we have $v(x) = A\exp(-\mu x) + B\exp(\mu x)$ for which the boundary conditions imply that $A = B = 0$.
3. For $\lambda = -\mu^2 < 0$, we have $v(x) = A\cos(\mu x) + B\sin(\mu x)$ for which the boundary conditionals imply that $A \in \mathbb{R}$, $B = 0$ and $\mu = \mu_n = \frac{\pi n}{L}$ for $n = 1, 2, \ldots$. Moreover, we have that $w_n(t) = C\exp(-\kappa\mu_n^2 t)$ where $C \in \mathbb{R}$.

□

From above we have obtained the family of solutions

$$u_n(t,x) = \exp\left(-\frac{\kappa n^2 \pi^2 t}{L^2}\right) \cos\left(\frac{\pi n x}{L}\right)$$

which we superimpose to define

$$u(t,x) = \sum_{n=0}^{\infty} A_n \exp\left(-\frac{\kappa n^2 \pi^2 t}{L^2}\right) \cos\left(\frac{\pi n x}{L}\right).$$

Here the A_n are arbitrary, however, if we fix them as stated in the theorem then we see that the initial condition $u(0,x) = g(x)$ is satisfied. This time we consider an even extension of g and construct its Fourier coefficients in the same way as before.

3.2.3 Uniqueness

We can approach the problem of uniqueness for the general diffusion equation

$$\begin{cases} \partial_t u - \kappa \Delta u = f & (t,\mathbf{x}) \in Q_T \\ u(0,\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \Omega \end{cases} \tag{3.2.3}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary $\partial\Omega$. We assume the boundary conditions of (3.2.3) are of either Dirichlet or Neumann type.

> **Theorem 3.2.3.** *The diffusion equation* (3.2.3) *with homogeneous Dirichlet or Neumann conditions has at most one solution belonging to the class* $\mathcal{C}_t^1\mathcal{C}_x^2\left(\bar{Q}_T\right)$.

Proof. Assume that u_1 and u_2 are functions in $\mathcal{C}_t^1\mathcal{C}_x^2\left(\bar{Q}_T\right)$ and solve (3.2.3). Let $w = u_1 - u_2$, then it follows that w satisfies

$$\begin{cases} \partial_t w - \kappa \Delta w = 0 & (t,\mathbf{x}) \in Q_T \\ w(0,\mathbf{x}) = 0 & \mathbf{x} \in \Omega, \end{cases} \tag{3.2.4}$$

with homogeneous boundary conditions of either Dirichlet or Neumann type. It follows that

$$\int_\Omega (\partial_t w - \kappa \Delta w) w \, \mathrm{d}\mathbf{x} = 0. \tag{3.2.5}$$

Using this we deduce that

$$\kappa \int_\Omega \Delta w w \, \mathrm{d}\mathbf{x} = \int_\Omega \partial_t w w \, \mathrm{d}\mathbf{x} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |w|^2 \, \mathrm{d}\mathbf{x}. \tag{3.2.6}$$

On the other hand, proceeding directly with integration by parts and using the boundary conditions we have

$$\begin{aligned} \kappa \int \Delta w w \, \mathrm{d}\mathbf{x} &= -\kappa \int_\Omega |\nabla w|^2 \, \mathrm{d}\mathbf{x} + \kappa \int_{\partial\Omega} \partial_{\mathbf{n}} w w \, \mathrm{d}\sigma \\ &= -\kappa \int_\Omega |\nabla w|^2 \, \mathrm{d}\mathbf{x}. \end{aligned} \tag{3.2.7}$$

Let

$$E(t) = \frac{1}{2}\int_\Omega |w|^2 \, \mathrm{d}\mathbf{x}.$$

Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(t) &\overset{(3.2.6)}{=} \int_\Omega \partial_t w w \, \mathrm{d}\mathbf{x} \\ &\overset{(3.2.5)}{=} \kappa \int_\Omega \Delta w w \, \mathrm{d}\mathbf{x} \\ &\overset{(3.2.7)}{=} -\kappa \int_\Omega |\nabla w|^2 \, \mathrm{d}\mathbf{x} \leq 0. \end{aligned}$$

Therefore, as $E(0) = 0$ by the boundary conditions it follows that $E(t) \leq 0$ for $t \in [0,T]$. This implies that $\nabla w = 0$ in Ω and so $u_1 = u_2$. $\square$

3.2.4 Maximum Principle

As heat flows from higher to lower temperatures, we know that a solution to a homogeneous heat equation must attain its maximum and minimum values on the boundary.

Theorem 3.2.4 (Weak Maximum Principle)**.** *Let $w \in \mathcal{C}_t^1\mathcal{C}_x^2(Q_T) \cap \mathcal{C}\left(\bar{Q}_T\right)$ be a function such that*

$$\partial_t w - \kappa \Delta w = q \leq 0$$

in Q_T. Then w attains its maximum on Q_T, that is

$$\max_{\bar{Q}_T} w = \max_{\partial Q_T} w.$$

Proof. For $\epsilon > 0$, let $u = w - \epsilon T$, then

$$\partial_t u - \kappa \Delta u = q - \epsilon < 0.$$

Assume that there exists a $(t_0, \mathbf{x}_0) \in (0, T - \epsilon] \times \Omega$ where u attains it maximum on $\bar{Q}_{T-\epsilon}$. It follows that $\Delta u(t_0, \mathbf{x}_0) \leq 0$ and

$$\begin{cases} \partial_t u(t_0, \mathbf{x}_0) = 0 & t_0 \in (0, T - \epsilon), \\ \partial_t u(t_0, x_0) \geq 0 & t_0 = T - \epsilon. \end{cases}$$

In either case

$$\partial_t u(t_0, \mathbf{x}_0) - \kappa \Delta u(t_0, \mathbf{x}_0) \geq 0,$$

which is a contradiction. Therefore,

$$\max_{\bar{Q}_{T-\epsilon}} u \leq \max_{\partial Q_T} u \leq \max_{\partial Q_T} w.$$

Therefore, as $w \leq u + \epsilon T$, we have

$$\max_{\bar{Q}_{T-\epsilon}} \leq \max_{\bar{Q}_{T-\epsilon}} (u + \epsilon T) \leq \max_{\partial Q_T} (w + \epsilon T). \tag{3.2.8}$$

Since w is continuous in $\bar{Q}_T$ we deduce that

$$\lim_{\epsilon} \max_{\bar{Q}_{T-\epsilon}} w = \max_{\bar{Q}_T} w.$$

Sending $\epsilon \to 0$ in (3.2.8) it follows that

$$\max_{\bar{Q}_T} w = \max_{\partial Q_T} w.$$

□

Remark 3.2.5.

1. *Note that in the context of Theorem 3.2.4 it is possible that the maximum, or minimum, is achieved at an interior point as well.*
2. *If u is a solution to the diffusion equation*

$$\partial_t u - \kappa \Delta u = 0$$

in Q_T, then u attains its maximum and minimum on ∂Q_T. That is,

$$\min_{\partial Q_T} \leq u(t, \mathbf{x}) \leq \max_{\partial Q_T} u$$

for all $(t, \mathbf{x}) \in \bar{Q}_T$.

3. *Suppose*
$$\partial_t u_1 - \kappa\Delta u_1 = f_1$$
and
$$\partial_t u_2 - \kappa\Delta u_2 = f_2.$$
Then using Theorem 3.2.4 we deduce the following.

   (a) *If $u_1 \geq u_2$ on ∂Q_T and $f_1 \geq f_2$ in Q_T, then $u_1 \geq u_2$ in Q_T.*

   (b) *The following stability estimate holds true,*
$$\max_{\bar{Q}_T} |u_1 - u_2| \leq \max_{\partial Q_T} |u_1 - u_2| + T \max_{\bar{Q}_T} |f_1 - f_2|.$$
   Thus if we are operating under Dirichlet initial conditions it follows that there is at most one solution that depends continuously on the data.

3.3 Problems on the Whole Space

3.3.1 The Fundamental Solution

Throughout this section, we will analyse the diffusion equation
$$\begin{cases} \partial_t u - \kappa\Delta u = f & (t, \mathbf{x}) \in (0, T) \times \mathbb{R}^d \\ u(0, \mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d. \end{cases} \tag{3.3.1}$$

Suppose that u is a solution to (3.3.1) with $f = 0$, that is
$$\partial_t u - \kappa\Delta u = 0 \tag{3.3.2}$$

In such a case, u will have the following properties.

1. Time Reversal. That is, $v(t, \mathbf{x}) = u(-t, \mathbf{x})$ where v is a satisfied the backward equation $\partial_t v + \kappa\Delta v = 0$.
2. Space and Time Translation Invariance. That is, for $\mathbf{y} \in \mathbb{R}^d$ and fixed $s \in \mathbb{R}$, the function $v(t, \mathbf{x}) = u(t - s, \mathbf{x} - \mathbf{y})$ still satisfies (3.3.2).
3. Parabolic Dilations. That is, for fixed $a > 0$, the function $v(t, \mathbf{x}) = u\left(a^2 t, a\mathbf{x}\right)$ still satisfies (3.3.2).

Exercise 3.3.1. *Using the properties exhibited by a solution to* (3.3.2)*, construct a solution to* (3.3.2) *for the case $d = 1$.*

Definition 3.3.2. *The function*
$$\Gamma_\kappa(t, \mathbf{x}) = \frac{1}{(4\pi\kappa t)^{\frac{d}{2}}} \exp\left(-\frac{|\mathbf{x}|^2}{4\kappa t}\right)$$
for $\mathbf{x} \in \mathbb{R}^d$ and $t > 0$ is called the fundamental solution of the heat equation (3.3.2)*.*

We can examine the behaviour of the fundamental solution for the case $d = 1$. Note that

- $\lim_{t \to 0^+} \Gamma_\kappa(t, x) = 0$ for $x \neq 0$, while
- $\lim_{t \to 0^+} \Gamma_\kappa(t, 0) = \infty$.

That is, the fundamental solution concentrates mass at the origin as $t \to 0^+$. To capture this behaviour rigorously we can utilise the Dirac measure.

Definition 3.3.3. *The Dirac measure at the origin is the generalized function, δ_0, that acts on a test function v, which is smooth function with compact support, as*

$$\langle \delta_0, v \rangle = \int_{\mathbb{R}} \delta_0(x) v(x) dx = v(0).$$

More generally, the Dirac measure at y acts on a test function v as

$$\langle \delta_y, v \rangle = v(y).$$

Remark 3.3.4. *Definition 3.3.3 holds equally in higher dimensions*

Using Definition 3.3.3, the initial condition satisfied by the fundamental solution can be written as

$$\Gamma_\kappa(0, \mathbf{x} - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}).$$

3.3.2 The Homogeneous Problem

Now consider the homogeneous diffusion equation

$$\begin{cases} \partial_t u - \kappa \Delta u = 0 & (t, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^d \\ u(0, \mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d \end{cases} \tag{3.3.3}$$

Theorem 3.3.5. *Let $g \in \mathcal{C}_b\left(\mathbb{R}^d\right)$, that is a continuous and bounded function. Then*

$$u(t, \mathbf{x}) = \int_{\mathbb{R}^d} \Gamma_\kappa(t, \mathbf{x} - \mathbf{y}) g(\mathbf{y}) \, \mathrm{d}\mathbf{y} = \frac{1}{(4\pi\kappa t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\left(-\frac{|\mathbf{x} - \mathbf{y}|^2}{4\kappa t}\right) g(\mathbf{y}) \, \mathrm{d}\mathbf{y}$$

defines an infinitely differentiable function on $(0, \infty) \times \mathbb{R}^d$ that satisfies

$$\partial_t u - \kappa \Delta u = 0$$

and

$$\lim_{t \to 0^+} u(t, \mathbf{x}) = g(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^d$. Moreover, u is the unique solution to (3.3.3).

Remark 3.3.6. *From Theorem 3.3.5 we observe that there is a smoothing effect. Namely, as $t > 0$ the initial data is regularised and the solution becomes infinitely differentiable. Moreover, if g is positive and compactly supported we have that u is everywhere positive for $t > 0$.*

3.3.3 The In-Homogeneous Problem

Having considered the homogeneous case, we return to the full problem stated in (3.1.1). Note how the solution, $u(t, \cdot)$, given in the above theorem is dependent on the initial datum $g \in C_b\left(\mathbb{R}^d\right)$. So by the uniqueness conclusion of the theorem, we can define an operator $S(t) : C_b\left(\mathbb{R}^d\right) \to C_b\left(\mathbb{R}^d\right)$ defined by

$$g \mapsto S(t)g = u(t, \cdot).$$

Note,

1. $S(t)$ is a convolution with the fundamental solution, and
2. that $S(0)$ is the identity.

Theorem 3.3.7. *Let $g \in \mathcal{C}_b\left(\mathbb{R}^d\right)$ and $f \in \mathcal{C}_b^2\left([0,\infty)\times\mathbb{R}^d\right)$. Then*

$$u(t,\mathbf{x}) = \int_{\mathbb{R}^d} \Gamma_k(t,\mathbf{x}-\mathbf{y})g(\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_0^t \int_{\mathbb{R}^d} \Gamma_k(t-s,\mathbf{X}-\mathbf{y})f(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s \tag{3.3.4}$$

is the unique solution to the non-homogeneous diffusion equation (3.1.1)*.*

Exercise 3.3.8. *Verify that* (3.3.4) *is indeed a solution to* (3.1.1)*.*

Remark 3.3.9. *Note how the solution of the above theorem is of the form*

$$u(t,\cdot) = S(t)g + \int_0^t S(t-s)f(s)ds.$$

3.4 Solution to Exercises

Exercise 3.3.1

Solution. Due to the invariance under the change of co-ordinates $(t,\mathbf{x}) \mapsto \left(a^2t, ax\right)$ we consider solutions of the form

$$u^\star(t,x) = \frac{1}{\sqrt{\kappa t}} U\left(\frac{x}{\sqrt{kt}}\right), \tag{3.4.1}$$

where for $\xi = \frac{x}{\sqrt{\kappa t}}$ we have $U(\xi) \overset{\xi\to\pm\infty}{\longrightarrow} 0$. Thus,

$$\partial_t u^\star = -\frac{1}{2\sqrt{kt^3}}\left(U + \frac{x}{\sqrt{\kappa t}}U'\right)$$

and

$$\partial_{xx}u^\star = \frac{1}{(\kappa t)^{\frac{3}{2}}}U''.$$

Therefore,

$$0 = -\frac{1}{\sqrt{\kappa t^3}}\left(\frac{1}{2}U + \frac{x}{2\sqrt{\kappa t}}U' + U''\right)$$

which implies that

$$U'' + \frac{\xi}{2}U' + \frac{1}{2}U = 0. \tag{3.4.2}$$

Now the time reversal manifests as an invariance to the change of variables $\xi \mapsto -\xi$. Which means that $U(\xi) = U(-\xi)$ and $U'(0) = 0$. Therefore, using (3.4.2) it follows that

$$\left(U' + \frac{\xi}{2}\right)' = 0$$

which implies that

$$U' + \frac{\xi}{2}U = 0.$$

Solving this we arrive at

$$U(\xi) = c_0 \exp\left(-\frac{\xi^2}{4}\right).$$

Where $c_0 \in \mathbb{R}$ is such that U has unit mass, that is

$$\int_{\mathbb{R}} U(\xi)\,\mathrm{d}\xi = 1$$

and so $c_0 = \frac{1}{\sqrt{4\pi}}$. Returning to (3.4.1) we get that

$$u^{\star}(t,x) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right)$$

for $x \in \mathbb{R}$ and $t > 0$. $\square$

Exercise 3.3.8

Solution. We can differentiate the second term of (3.3.4), and recall that $u = \Gamma_\kappa(t, \mathbf{x} - \mathbf{y})$ solves

$$\partial_t u - \kappa \Delta u = 0$$

to find that

$$\partial_t \int_0^t \int_{\mathbb{R}^d} \Gamma_k(t-s, \mathbf{x}-\mathbf{y}) f(s, \mathbf{y}\, \mathrm{d}\mathbf{y}\, \mathrm{d}s = f(t,\mathbf{x}) + \int_0^t \partial_t \int_{\mathbb{R}^d} \Gamma_\kappa(t-s, \mathbf{x}-\mathbf{y}) f(s,\mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}s. \tag{3.4.3}$$

Note that $\int_{\mathbb{R}^d} \Gamma_k(t - s\mathbf{x} - \mathbf{y}) f(s, \mathbf{y})\, \mathrm{d}\mathbf{y}$ is the solution to the time-translated homogeneous heat equation with initial data $f(s, \mathbf{x})$, therefore,

$$(\partial_t - \kappa\Delta) \int_{\mathbb{R}^d} \Gamma_k(t - s\mathbf{x} - \mathbf{y}) f(s,\mathbf{y})\, \mathrm{d}\mathbf{y} = 0.$$

Thus, using (3.4.3) we get that

$$(\partial_t - \kappa\Delta) \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma_\kappa(t-s, \mathbf{x}-\mathbf{y}) f(s,\mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}s \tag{3.4.4}$$

$$= f(t,\mathbf{x}) + \int_0^t \partial_t \int_{\mathbb{R}^d} \Gamma_\kappa(t-s, \mathbf{x}-\mathbf{y}) f(s,\mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}s$$

$$= f(t,\mathbf{x}). \tag{3.4.5}$$

We know that

$$\int_{\mathbb{R}^d} \Gamma_\kappa(t, \mathbf{x}-\mathbf{y}) g(\mathbf{y})\, \mathrm{d}\mathbf{y}$$

solves the homogeneous problem and so

$$(\partial_t - \kappa\Delta) \int_{\mathbb{R}^d} \Gamma_\kappa(t, \mathbf{x}-\mathbf{y}) g(\mathbf{y})\, \mathrm{d}\mathbf{y} = 0. \tag{3.4.6}$$

Using (3.4.6) and (3.4.4) it is clear that

$$\int_{\mathbb{R}^d} \Gamma_k(t, \mathbf{x}-\mathbf{y}) g(\mathbf{y})\, \mathrm{d}\mathbf{y} + \int_0^t \int_{\mathbb{R}^d} \Gamma_k(t-s, \mathbf{X}-\mathbf{y}) f(s,\mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}s = f(t,\mathbf{x})$$

and so solves (3.1.1). $\square$

4 The Wave Equation

4.1 The One-Dimensional Wave Equations

We now construct a classical model for vibrations on a tightly stretched string using the following assumptions.

1. Vibrations are of relatively small amplitude.
2. Points on the string only undergo vertical displacements.
3. The vertical displacement of a point depends only on time and its position on the string.
4. The string is flexible.
5. Friction is negligible.

To work under these assumptions we make the following notational definitions.

1. $\rho_0 = \rho_0(x)$ is the linear density of the string at rest, and $\rho = \rho(t, x)$ is the density at time t.
2. $\tau(t, x)$ denotes the magnitude of the tension at x at time t.
3. $f(t, x)$ denotes the magnitude of the vertical body forces per unit mass.

Using Newton's law and the conservation of mass we deduce that

$$\partial_{tt} u - c^2(t, x)\partial_{xx} u = f, \tag{4.1.1}$$

where $c^2 = \frac{\tau_0(t)}{\rho_0(x)}$. The equation (4.1.1) is known as the one-dimensional wave equation. We can also consider the implications of the conservation of energy under the added assumptions that the string is elastic and has length L.

1. The total kinetic energy due to the vibrations is given by
$$E_{\mathsf{kin}}(t) = \frac{1}{2}\int_0^L \rho|\partial_t u(t, x)|^2 \, \mathrm{d}x.$$
2. The potential energy is given by
$$E_{\mathsf{pot}}(t) = \frac{1}{2}\int_0^L \tau_0|\partial_x u(t, x)|^2 \, \mathrm{d}x.$$
3. The total energy is given by
$$E(t) = E_{\mathsf{kin}}(t) + E_{\mathsf{pot}}(t).$$

If $f = 0$, and $\partial_t u(t, L) = \partial_t(t, 0) = 0$, then $\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0$ and so energy is conserved.

Condition	Informal Definition	Formal Definition
Dirichlet	The displacement of the endpoints is known.	$u(t, 0) = a(t)$ and $u(t, L) = b(t)$ for $t \in (0, T]$.
Neumann	The tension applied to the endpoints is known.	$\partial_x u(t, 0) = a(t)$ and $\partial_x u(t, L) = b(t)$ for $t \in (0, T]$.
Robin	Describes a linear elastic attachment to the endpoints.	$\tau_0\partial_x u(t, 0) = ku(t, 0)$ and $\tau_0\partial_x u(t, L) = -ku(t, L)$ for $k > 0$ the elastic constant.

4.1.1 One-Dimensional Dirichlet Problem

Consider the Cauchy-Dirichlet problem

$$\begin{cases} \partial_{tt}u - c^2\partial_{xx}u = 0 & (t,x) \in (0,T) \times (0,L) \\ u(t,0) = u(t,L) = 0 & t \in (0,T) \\ u(0,x) = g(x), \quad \partial_t u(0,x) = h(x) & x \in [0,L] \end{cases} \tag{4.1.2}$$

where c is a constant.

Theorem 4.1.1. *Let $g \in \mathcal{C}^4([0,L])$ and $h \in \mathcal{C}^3([0,L])$ with $g(0) = g(L) = g''(0) = g'' = 0$ and $h(0) = h(L) = 0$. Then the function*

$$u(t,x) = \sum_{n=1}^{\infty} \left(\hat{g}_n \cos(\mu_n ct) + \frac{\hat{h}_n}{\mu_n c} \sin(\mu_n ct) \right) \sin(\mu_n),$$

is the unique solution to (4.1.2).

4.1.2 The d'Alembert Formula

Consider the global Cauchy problem

$$\begin{cases} \partial_{tt}u - c^2\partial_{xx}u = 0 & (t,x) \in (0,\infty) \times \mathbb{R} \\ u(0,x) = g(x), \quad \partial_t u(0,x) = h(x) & x \in \mathbb{R}. \end{cases} \tag{4.1.3}$$

Under the assumption that $g(x) \in \mathcal{C}^2(\mathbb{R})$ and $h \in \mathcal{C}^1(\mathbb{R})$, and imposing the initial condition $g(x) = u(0,x)$, we get that

$$u(t,x) = \frac{1}{2}(g(x+ct) - g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,\mathrm{d}y \tag{4.1.4}$$

is a unique $\mathcal{C}^2$-solution to (4.1.3) in the half-plane $[0,\infty) \times \mathbb{R}$. This solution (4.1.4) is known as the d'Alembert formula and one can re-write this solution as

$$u(t,x) = F(x+ct) + G(x-ct)$$

where

- $F(x+ct) = \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_0^{x+ct} h(y)\,\mathrm{d}y$, and
- $G(x-ct) = \frac{1}{2}g(x-ct) + \frac{1}{2}\int_{x-ct}^0 h(y)\,\mathrm{d}y$.

This re-writing highlights the characteristics $x + ct = \text{const}$ and $x - ct = \text{const}$. The domain of dependence of (t,x) is the domain on which the value of $u(t,x)$ is dependent on, which is $g(x+ct)$, $g(x-ct)$, and $h(x)$ for $x \in [x-ct, x+ct]$. Similarly, values of g and h at a point z influence $u(t,x)$ for

$$z - ct \leq x \leq z + ct,$$

which is called the range of influence of z.

Example 4.1.2. *Consider*

$$\begin{cases} \partial_{tt}u - c^2\partial_{xx}u = 0 & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = g(x) & x \in \mathbb{R}, \\ \partial_t u(0,x) = 0 & x \in \mathbb{R}, \end{cases}$$

where

$$g(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

for some $a > 0$. The solution to this problem given by (4.1.4) is

$$u(t,x) = \frac{1}{2}(g(x+ct) + g(x-ct)).$$

There are different cases one now has to consider.

1. *If $x > a + ct$, then $x > a - ct$. Hence, $g(x-ct) = g(x+ct) = 0$and so*
$$u(x,t) = 0.$$
2. *If $\max(a-ct, -a+ct) < x < a+ct$, then $g(x-ct) = 1$, $g(x+ct) = 0$ and so*
$$u(t,x) = \frac{1}{2}.$$
3. *If $\min(a-ct, -a+ct) < x < \max(a-ct, -a+ct)$, then $g(x-ct) = g(x+ct) = 1$ and so*
$$u(t,x) = 1.$$
4. *If $-a+ct < x < a-ct$, then $g(x-ct) = g(x+ct) = 1$ and so*
$$u(t,x) = 1.$$
5. *If $a-ct < x < -a+ct$, then $g(x-ct) = g(x+ct) = 0$ and so*
$$u(t,x) = 0.$$
6. *If $-a-ct < x < \min(a-ct, -a+ct)$, then $g(x-ct) = 0$, $g(x+ct) = 1$ and so*
$$u(t,x) = \frac{1}{2}.$$
7. *If $x < -a-ct$, then $x < -a+ct$. Hence, $g(x-ct) = g(x+ct) = 0$ and so*
$$u(t,x) = 0.$$

4.2 The Wave Equation in Three-Dimensions

Consider the homogeneous global Cauchy problem for the wave equation

$$\begin{cases} \partial_{tt}u - c^2\Delta u = 0 & (t,\mathbf{x}) \in (0,\infty)\times\mathbb{R}^d \\ u(0,\mathbf{x}) = g(\mathbf{x}) & x \in \mathbb{R}^d, \\ \partial_t u(0,\mathbf{x}) = h(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d, \end{cases} \tag{4.2.1}$$

where $d = 3$.

4.2.1 Huygens Principle

Let w_h be the solution to the problem (4.2.1) where $g(\mathbf{x}) = 0$ and $d = 3$.

Lemma 4.2.1. *If w_g has continuous third-order partial derivatives, then $v = \partial_t w_g$ solves (4.2.1) with $h(\mathbf{x}) = 0$. In which case, the solution of (4.2.1) is given by*

$$u = \partial_t w_g + w_h.$$

Definition 4.2.2. *For some $\mathbf{y} \in \mathbb{R}^3$, we call the solution of* (4.2.1) *with $d = 3$, $g(\mathbf{x}) = 0$ and $h(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y})$ the fundamental solution of the three-dimensional wave equation and denote it $K(t, \mathbf{x}, \mathbf{y})$.*

Proposition 4.2.3. *The fundamental solution of the three-dimensional wave equation is given by*

$$K(t, \mathbf{x}, \mathbf{y}) = \frac{\delta(r - ct)}{4\pi cr}$$

where $r = |\mathbf{x} - \mathbf{y}|$.

Proof. Recall that

$$\Gamma(\epsilon, \mathbf{x} - \mathbf{y}) = \frac{1}{(4\pi\epsilon)^{\frac{3}{2}}} \exp\left(-\frac{|\mathbf{x} - \mathbf{y}|^2}{4\epsilon}\right) \to \delta(\mathbf{x} - \mathbf{y})$$

as $\epsilon \to 0$. Thus, we attempt to approximate solutions to (4.2.1) by replacing $\delta(\mathbf{x} - \mathbf{y})$ with $\Gamma(\epsilon, \mathbf{x} - \mathbf{y})$. Indeed let w_ϵ be the solution of (4.2.1) with $\delta(\mathbf{x} - \mathbf{y})$ replaced by $\Gamma(\epsilon, \mathbf{x} - \mathbf{y})$. As $\Gamma(\epsilon, \mathbf{x} - \mathbf{y})$ is radially symmetric with a pole at $\mathbf{y}$ we expect that $w_\epsilon = w_\epsilon(t, r)$ for $r = |\mathbf{x} - \mathbf{y}|$. Passing the equation of (4.2.1) to spherical coordinates it follows that

$$\partial_{tt} w_\epsilon - c^2 \left(\partial_{rr} + \frac{2}{r}\partial_r\right) w_\epsilon = 0$$

which happens if and only if

$$\partial_{tt}(rw_\epsilon) - c^2 \partial_{rr}(rw_\epsilon) = 0.$$

In particular, rw_ϵ satisfies the one-dimensional wave equation and thus using (4.1.4) we have

$$w_\epsilon(t, r) = \frac{1}{r}\left(F(r + ct) + g(r - ct)\right)$$

where F and G are to be determined. From the initial conditions we have

$$\begin{cases} F(r) + G(r) = 0, \\ c\left(F'(r) - G'(r)\right) = r\Gamma(\epsilon, r). \end{cases}$$

Equivalently,

$$\begin{cases} F(r) = -G(r), \\ G'(r) = -\frac{r\Gamma(\epsilon, r)}{2c}. \end{cases}$$

Therefore,

$$G(r) = \frac{1}{4\pi c\sqrt{4\pi\epsilon}}\left(\exp\left(-\frac{r^2}{4\epsilon}\right) - 1\right)$$

and so

$$w_\epsilon(t, r) = \frac{1}{4\pi c\sqrt{4\pi\epsilon} r}\left(\exp\left(-\frac{(r - ct)^2}{4\epsilon}\right) - \exp\left(\frac{(r + ct)^2}{4\epsilon}\right)\right).$$

As $\Gamma(\epsilon, r)$ is the fundamental solution of the fundamental diffusion equation with $x = r$ and $t = \epsilon$, it follows that

$$w_\epsilon(t, r) \xrightarrow{\epsilon \to 0} \frac{1}{4\pi cr}\left(\delta(r - ct) - \delta(r + ct)\right).$$

As $r + ct > 0$ for $t > 0$ we conclude that

$$K(t, \mathbf{x}, \mathbf{y}) = \frac{\delta(r - ct)}{4\pi cr}$$

where $r = |\mathbf{x} - \mathbf{y}|$. □

From Proposition 4.2.3, we see that the fundamental solution is a travelling wave, propagated from the point $\mathbf{y}$ to the region

$$\partial B_{ct}(\mathbf{y}) = \{\mathbf{x} : |\mathbf{x} - \mathbf{y}| = ct\}$$

which is called the support of K. This coincides with the forward space-time cone

$$C^*_{0,\mathbf{y}} = \{(t, \mathbf{x}) : |\mathbf{x} - \mathbf{y}| \leq ct, t > 0\}$$

from which we can define the range of influence of $\mathbf{y}$ as $\partial C^*_{0,\mathbf{y}}$. This is the strong Huygens Principle which explains that perturbations at $\mathbf{y}$ are only felt as sharp signals at a point $\mathbf{x}_0$ at a time $t_0 = \frac{\mathbf{x}_0 - \mathbf{y}}{c}$ after the perturbation.

4.2.2 The Kirchhoff Formula

We can exploit the work of the previous section to find a solution to the general form of (4.2.1). More specifically we note that

$$h(\mathbf{x}) = \int_{\mathbb{R}^3} \delta(\mathbf{x} - \mathbf{y}) h(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

That is, the solution to (4.2.1) can be seen as the superposition of the corresponding solutions $K(t, \mathbf{X}, \mathbf{y})h(\mathbf{y})$, more specifically,

$$w_h(t, \mathbf{x}) = \int_{\mathbb{R}^3} K(t, \mathbf{x}, \mathbf{y}) h(\mathbf{y}) \, \mathrm{d}\mathbf{y} = \frac{1}{4\pi c^2 t} \int_{\partial B_{\mathrm{ct}}(\mathbf{x})} h(\sigma) \, \mathrm{d}\sigma.$$

Theorem 4.2.4 (Kirchhoff Formula). *Let $g \in \mathcal{C}^3\left(\mathbb{R}^3\right)$ and $h \in \mathcal{C}^2\left(\mathbb{R}^3\right)$, then*

$$u(t, \mathbf{x}) = \partial_t \left(\frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\mathbf{x})} g(\sigma) d\sigma \right) + \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\mathbf{x})} h(\sigma) d\sigma$$

is the unique $\mathcal{C}^2\left([0, \infty) \times \mathbb{R}^3\right)$ solution to (4.2.1).

Remark 4.2.5. *We also have the alternative form of Kirchhoff formula given by*

$$u(t, \mathbf{x}) = \frac{1}{4\pi c^2 t^2} \int_{\partial B_{ct}(\mathbf{x})} g(\sigma) + \nabla g(\sigma) \cdot (\sigma - \mathbf{x}) + t h(\sigma) d\sigma.$$

From this form we can deduce that for g and h having compact support D, then a disturbance localized in D starts affecting $\mathbf{x}$ at time $t_{\min}$ and stops affect $\mathbf{x}$ at time $t_{\max}$. Where $t_{\min}$ and $t_{\max}$ are the first and last times that $D \cap \partial B_{ct}(\mathbf{x}) \neq \emptyset$. Therefore, Huygens's principle still applies.

4.3 The Wave Equation in Two-Dimensions

Consider the homogeneous global Cauchy problem for the wave equation

$$\begin{cases} \partial_{tt} u - c^2 \Delta u = 0 & (t, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^d \\ u(0, \mathbf{x}) = g(\mathbf{x}) & x \in \mathbb{R}^d \\ \partial_t u(0, \mathbf{x}) = h(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d, \end{cases} \tag{4.3.1}$$

where $d = 2$. One can apply the work done in three-dimensions to two-dimension by writing $\tilde{\mathbf{x}} \in \mathbb{R}^3$ as $(\mathbf{x}, x_3)$ for $\mathbf{x} \in \mathbb{R}^2$. This approach is known as the Hadamard method of descent.

Theorem 4.3.1 (Poisson Formula). *Let $g \in \mathcal{C}^3\left(\mathbb{R}^2\right)$ and $h \in \mathcal{C}^2\left(\mathbb{R}^2\right)$. Then*

$$u(t, \mathbf{x}) = \frac{1}{2\pi c} \left(\partial_t \int_{B_{ct}(\mathbf{x})} \frac{g(\mathbf{y})}{c^2 t^2 - |\mathbf{x} - \mathbf{y}|^2} \, \mathrm{d}\mathbf{y} + \int_{B_{ct}(\mathbf{x})} \frac{h(\mathbf{y})}{c^2 t^2 - |\mathbf{x} - \mathbf{y}|^2} \, \mathrm{d}\mathbf{y} \right)$$

is the unique $\mathcal{C}^2\left([0,\infty)\times\mathbb{R}^2\right)$ solution to $(4.3.1)$ with $d=2$.

Remark 4.3.2. *The solution given by the above theorem can be re-written as*

$$u(t,\mathbf{x}) = \frac{1}{2\pi ct}\int_{B_{ct}(\mathbf{x})} \frac{g(\mathbf{y}) + \nabla g(\mathbf{y})\cdot(\mathbf{y}-\mathbf{x}) + th(\mathbf{y})}{c^2t^2 - |\mathbf{x}-\mathbf{y}|^2}\,\mathrm{d}\mathbf{y}.$$

From which we observe that the domain of dependence for $(t,\mathbf{x})$ is given by

$$B_{ct}(\mathbf{x}) = \{\mathbf{y} : |\mathbf{x}-\mathbf{y}| < ct\}.$$

Interestingly, this shows a perturbation localized at ξ affects $\mathbf{x}$ at $t_{\min} = \frac{|\mathbf{x}-\xi|}{c}$, however, the disturbance remains for $t > t_{\min}$, and so Huygens principle does not apply in two-dimensions.

5 The Laplace Equation

5.1 The Laplace and Poisson Equation

The equation

$$\Delta u = 0$$

is known as the Laplace equation. More generally, for $\Omega \subset \mathbb{R}^d$ a bounded domain, the equation

$$\Delta u = f \tag{5.1.1}$$

in Ω is known as the Poisson equation. There are no initial conditions to this problem, however, we can impose boundary conditions.

Condition	Formal Definition
Dirichlet	$u(\sigma) = h(\sigma)$ for $\sigma \in \partial\Omega$ and some function h.
Neumann	$\partial_{\mathbf{n}} u(\sigma) = h(\sigma)$ where $\sigma \in \partial\Omega$, $\mathbf{n}$ is the outward normal to $\partial\Omega$ and h is some function.
Robin	$\partial_{\mathbf{n}} u(\sigma) + \alpha u(\sigma) = h(\sigma)$ where $\sigma \in \partial\Omega$, $\alpha > 0$, $\mathbf{n}$ is the outward normal to $\partial\Omega$ and h is some function.

Theorem 5.1.1. *Let $\Omega \subset \mathbb{R}^d$ be a smooth, bounded domain. Then there exists at most one solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1\left(\bar{\Omega}\right)$ to (5.1.1), under Dirichlet or Robin conditions. For Neumann conditions, two solutions that differ by a constant exist.*

Proof. Let u and v be solutions to (5.1.1) imposed with the same boundary conditions. Let $w = u - v$. Then w satisfies the corresponding homogeneous boundary condition. Consequently,

$$0 = \int_\Omega w\Delta w = \int_{\partial\Omega} w\partial_{\mathbf{n}} w - \int |\nabla w|^2$$

which implies that

$$\int_\Omega |\nabla w|^2 = \int_{\partial\Omega} w\partial_{\mathbf{n}} w.$$

Under Dirichlet or Neumann conditions we have

$$\int_{\partial\Omega} w\partial_{\mathbf{n}} w = 0,$$

and under Robin conditions we have

$$\int_{\partial\Omega} w\partial_{\mathbf{n}} w = -\alpha \int_{\partial\Omega} |w| \leq 0.$$

In any case, it follows that

$$\int_\Omega |\nabla w|^2 \leq 0$$

which implies that $\nabla w \equiv 0$and so $w = u - v = c$ for some $c \in \mathbb{R}$. Under the Dirichlet and Robin conditions, it must be the case that $c = 0$, however, under Neumann conditions we do not have such a restriction. ☐

Remark 5.1.2. *Note that for the Neumann boundary conditions to admit a solution, it has to satisfy the compatibility condition*

$$\int_\Omega f = \int_{\partial\Omega} h.$$

5.1.1 Harmonic Functions

Definition 5.1.3. *For a domain $\Omega \subset \mathbb{R}^d$, a function $u \in \mathcal{C}^2(\Omega)$ is harmonic if*

$$\Delta u = 0$$

in Ω.

Theorem 5.1.4. *Let u be harmonic in $\Omega \subset \mathbb{R}^d$, then for any ball $\overline{B_R(\mathbf{x})} \subset \Omega$ we have the mean value formula,*

$$u(\mathbf{x}) = \frac{d}{\omega_d R^d} \int_{B_R(\mathbf{x})} u(\mathbf{y})\,\mathrm{d}\mathbf{y} = \frac{1}{\omega_d R^{d-1}} \int_{\partial B_r(\mathbf{x})} u(\sigma)\,\mathrm{d}\sigma \tag{5.1.2}$$

where ω_d is the measure of $\partial B_1(\mathbf{0})$.

Proof. For simplicity, we suppose that $d = 2$, so that $\omega_2 = 2\pi$. Starting with the second integral of (5.1.2) we consider

$$\phi(r) = \frac{1}{2\pi r} \int_{\partial B_r(\mathbf{x})} u(\sigma)\,\mathrm{d}\sigma$$

for $r < R$. Letting $\sigma = \mathbf{x} + r\sigma'$ such that $\sigma' \in \partial B_1(\mathbf{0})$ and $\mathrm{d}\sigma = r\mathrm{d}\sigma'$ it follows that

$$\phi(r) = \frac{1}{2\pi} \int_{\partial B_1(\mathbf{0})} u\left(\mathbf{x} + r\sigma'\right)\,\mathrm{d}\sigma'.$$

Letting $v(\mathbf{y}) = u(\mathbf{x} + r\mathbf{y})$ we have that

$$\nabla v(\mathbf{y}) = r\nabla u(\mathbf{x} + r\mathbf{y})$$

and

$$\Delta v(\mathbf{y}) = r^2 \Delta u(\mathbf{x} + r\mathbf{y}).$$

Consequently,

$$\begin{aligned}
\phi'(r) &= \frac{1}{2\pi} \int_{\partial B_1(\mathbf{0})} \partial_r u\left(\mathbf{x} + r\sigma'\right)\,\mathrm{d}\sigma' \\
&= \frac{1}{2\pi} \int_{\partial B_1(\mathbf{0})} \nabla u\left(\mathbf{x} + r\sigma'\right) \cdot \sigma'\,\mathrm{d}\sigma'. \\
&= \frac{1}{2\pi r} \int_{\partial B_1(\mathbf{0})} \nabla v\left(\sigma'\right) \dot{\sigma}'\,\mathrm{d}\sigma' \\
&= \frac{1}{2\pi r} \int_{B_1(\mathbf{0})} \Delta v(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= \frac{r}{2\pi} \int_{B_1(\mathbf{0})} \Delta u(\mathbf{x} + r\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= 0.
\end{aligned}$$

Therefore, $\phi(r) = c$ for some $c \in \mathbb{R}$. As $\phi(r) \to u(\mathbf{x})$ as $r \to 0$ it must be the case that $c = u(\mathbf{x})$ and so

$$u(\mathbf{x}) = \frac{1}{\omega_2 r^2} \int_{\partial B_r(\mathbf{x})} u(\sigma)\,\mathrm{d}\sigma.$$

for all $r < R$. We can extend this to the second integral of (5.1.2) using continuity. Using this we can obtain the first integral by noting that

$$\frac{R^2}{2} u(\mathbf{x}) = \frac{1}{2\pi} \int_0^R \left(\int_{\partial B_r(\mathbf{x})} u(\sigma)\,\mathrm{d}\sigma \right)\,\mathrm{d}r = \frac{1}{2\pi} \int_{B_R(\mathbf{x})} u(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

$\square$

Definition 5.1.5. *A continuous function has the mean value property in Ω if* (5.1.2) *holds for all $B_R(\mathbf{x})$.*

Theorem 5.1.6. *For $u \in \mathcal{C}(\Omega)$, if u has the mean value property then $\mathcal{C} \in C^\infty(\Omega)$ and it is harmonic in Ω.*

Theorem 5.1.7. *For a domain $\Omega \subset \mathbb{R}^d$, let $u \in \mathcal{C}(\Omega)$. If u has the mean value property and attains its maximum or minimum at $\mathbf{p} \in \Omega$, the u is constant. If Ω is bounded and $u \in \mathcal{C}\left(\bar{\Omega}\right)$ is not constant, then for every $\mathbf{x} \in \Omega$ we have that*

$$u(\mathbf{x}) < \max_{\partial\Omega}(u)$$

and

$$u(\mathbf{x}) > \min_{\partial\Omega}(u).$$

Proof. For simplicity, we deal with the case when $d = 2$, and $\mathbf{p} \in \Omega$ is a minimum point of u. Thus

$$m : u(\mathbf{p}) \leq u(\mathbf{y})$$

for all $\mathbf{y} \in \Omega$. Let $\mathbf{q} \in \Omega$. As Ω is path connected there exists a sequence of balls $(B(\mathbf{x}_j))_{j=0,\ldots,N} \subseteq \Omega$ such that $x_j \in B(\mathbf{x}_{j-1})$ for $j = 1, \ldots, N$ with $\mathbf{x}_0 = \mathbf{p}$ and $\mathbf{x}_N = \mathbf{q}$. By the mean value property we have that

$$u(\mathbf{p}) = \frac{1}{|B(\mathbf{p})|} \int_{B(\mathbf{p})} u(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

Suppose that there exists a $\mathbf{z} \in B(\mathbf{p})$ such that $u(\mathbf{z}) > m$. As $B(\mathbf{p})$ there exists an $r > 0$ such that $B_r(\mathbf{z}) \subset B(\mathbf{p})$. Thus we can write

$$m = u(\mathbf{p}) = \frac{1}{|B(\mathbf{p})|} \left(\int_{B(\mathbf{p}) \setminus B_r(\mathbf{z})} u(\mathbf{y}) \, \mathrm{d}\mathbf{y} + \int_{B_r(\mathbf{z})} u(\mathbf{y}) \, \mathrm{d}\mathbf{y} \right).$$

Using the mean value property we also have that

$$\int_{B_r(\mathbf{z})} u(\mathbf{y}) \, \mathrm{d}\mathbf{y} = |B_r(\mathbf{z})| u(\mathbf{z}) > m |B_r(\mathbf{z})|.$$

Therefore, using the fact that $u(\mathbf{y}) \geq m$ for every $\mathbf{y}$ it follows that

$$\begin{aligned} m &\geq \frac{1}{|B(\mathbf{p})|} \left(m |B(\mathbf{p}) \setminus B_r(\mathbf{z})| + \int_{B_r(\mathbf{z})} u(\mathbf{y}) \, \mathrm{d}\mathbf{y} \right) \\ &> \frac{1}{|B(\mathbf{p})|} \left(m |B(\mathbf{p}) \setminus B_r(\mathbf{z})| + m |B_r(\mathbf{z})| \right) \\ &= m. \end{aligned}$$

Which is a contradiction and so no such $\mathbf{z}$ exists. Thus, it must be the case that $u \equiv m$ in $B(\mathbf{p})$. Hence, due to our sequence of balls it follows that $u(\mathbf{q}) = m$, from which it follows that $u \equiv m$ in Ω as $\mathbf{q} \in \Omega$ was arbitrary. ☐

Corollary 5.1.8. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $h \in \mathcal{C}(\partial\Omega)$. The problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$$

has at most one solution solution $u_h \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\bar{\Omega}\right)$. Let u_{h_1} and u_{h_2} be the solutions corresponding to the data $h_1, h_2 \in \mathcal{C}(\partial\Omega)$.

1. *If $h_1 \geq h_2$ on $\partial\Omega$ and $h_1 \neq h_2$, then $u_{h_1} > u_{h_2}$ in Ω.*

2. *The estimate*

$$|u_{h_1}(\mathbf{x}) - u_{h_2}(\mathbf{x})| \leq \max_{\partial\Omega} |h_1 - h_2|$$

holds for all $\mathbf{x} \in \Omega$.

Proof. Let $w = u_{h_1} - u_{h_2}$. Then w is harmonic and $w = h_1 - h_2 > 0$ on $\partial\Omega$. It follows from Theorem 5.1.7 that

$$w(\mathbf{x}) > \min_{\partial\Omega}(h_1 - h_2) \geq 0$$

for every $\mathbf{x} \in \Omega$. Hence, we arrive at statement 1. Applying Theorem 5.1.7 additionally to $-w$ we deduce that

$$\pm w(\mathbf{x}) < \max_{\partial\Omega} |h_1 - h_2|$$

for every $x \in \Omega$, which is equivalent to statement 2. □

5.1.2 The Dirichlet Problem on a Circle

Theorem 5.1.9 (Poisson Formula)**.** *For* $\mathbf{p} \in \mathbb{R}^d$ *and* $h \in \mathcal{C}(\partial B_R(\mathbf{p}))$*, the unique solution* $u \in \mathcal{C}^2(B_R(\mathbf{p})) \cap \mathcal{C}(B_R(\mathbf{p}))$ *of*

$$\begin{cases} \Delta u = 0 & \text{in } B_R(\mathbf{p}) \\ u = h & \text{on } \partial B_R(\mathbf{p}) \end{cases}$$

is given by the Poisson formula

$$u(\mathbf{x}) = \frac{R^2 - |\mathbf{x} - \mathbf{p}|^2}{\omega_d R} \int_{\partial B_R(\mathbf{p})} \frac{h(\sigma)}{|\mathbf{x} - \sigma|^d} \, \mathrm{d}\sigma,$$

where ω_d *is the measure of* $B_R(\mathbf{p})$*. Moreover,* $u \in \mathcal{C}^\infty(B_R(\mathbf{p}))$*.*

Remark 5.1.10. *In two dimensions and using polar coordinates,* $\sigma = R(\cos(\alpha), \sin(\alpha))$ *and* $\mathrm{d}\sigma = R\,\mathrm{d}\alpha$*, the Poisson formula can be written as*

$$u(r, \theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\alpha)}{R^2 + r^2 - 2Rr\cos(\theta - \alpha)} \, \mathrm{d}\alpha.$$

5.1.3 Harnack Inequality

Theorem 5.1.11 (Harnack Inequality)**.** *Let* u *be harmonic and non-negative in* $B_R(\mathbf{0}) \subset \mathbb{R}^d$*. Then*

$$\frac{R^{d-2}(R - |\mathbf{x}|)}{(R + |\mathbf{x}|)^{d-1}} u(\mathbf{0}) \leq u(\mathbf{x}) \leq \frac{R^{d-2}(R + |\mathbf{x}|)}{(R - |\mathbf{x}|)^{d-1}} u(\mathbf{0}).$$

Proof. For the proof, we consider the case when $d = 3$. When $B_R = B_R(\mathbf{p})$ is a 3-dimensional ball the Poisson formula of Theorem 5.1.9 can be written as

$$u(\mathbf{x}) = \frac{R^2 - |\mathbf{x}|^2}{\omega_3 R} \int_{\partial B_R} \frac{u(\sigma)}{|\mathbf{x} - \sigma|^3} \, \mathrm{d}\sigma.$$

Note that

$$R - |\mathbf{x}| \leq |\sigma - \mathbf{x}| \leq R + |\mathbf{x}|$$

and

$$R^2 - |\mathbf{x}|^2 = (R - |\mathbf{x}|)(R + |\mathbf{x}|).$$

Using the mean value property it follows that

$$u(\mathbf{x}) \leq \frac{R+|\mathbf{x}|}{(R-|\mathbf{x}|)^2}\frac{1}{4\pi R}\int_{B_r}\int_{\partial B_R} u(\sigma)\,\mathrm{d}\sigma = \frac{R(R+|\mathbf{x}|)}{(R-|\mathbf{x}|)^2}u(\mathbf{0}),$$

and

$$u(\mathbf{x}) \geq \frac{R-|\mathbf{x}|}{(R+|\mathbf{x}|)^2}\frac{1}{4\pi R}\int_{B_r}\int_{\partial B_R} u(\sigma)\,\mathrm{d}\sigma = \frac{R(R-|\mathbf{x}|)}{(R+|\mathbf{x}|)^2}u(\mathbf{0})$$

which completes the proof. □

Corollary 5.1.12 (Liouville Theorem)**.** *If u is harmonic in $\mathbb{R}^d$ and $u(\mathbf{x}) \geq M$, then u is constant.*

Proof. Note that $w = u - M$ is a non-negative harmonic function in $\mathbb{R}^d$. Let $\mathbf{x} \in \mathbb{R}^d$ and choose $R > |\mathbf{x}|$. It follows by Theorem 5.1.11 that

$$\frac{R^{d-2}(R-|\mathbf{x}|)}{(R+|\mathbf{x}|)^{d-1}}w(\mathbf{0}) \leq w(\mathbf{x}) \leq \frac{R^{d-2}(R+|\mathbf{x}|)}{(R-|\mathbf{x}|)^{d-1}}w(\mathbf{0}).$$

Letting $R \to \infty$ we deduce that

$$w(\mathbf{0}) \leq w(\mathbf{x}) \leq w(\mathbf{0}),$$

which implies that $w(\mathbf{x}) \equiv w(\mathbf{0})$ for any $\mathbf{x} \in \mathbb{R}^d$. □

5.1.4 Dirichlet Energy

Consider a smooth bounded domain $\Omega \subset \mathbb{R}^d$ and the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = h & \text{on } \partial\Omega. \end{cases} \tag{5.1.3}$$

Definition 5.1.13. *For $w \in C^2(\Omega) \cap \mathcal{C}\left(\bar{\Omega}\right)$, the Dirichlet energy of w is*

$$E(w) = \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x}.$$

Theorem 5.1.14. *A function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\bar{\Omega}\right)$ solves* (5.1.3) *if and only if it minimizes the Dirichlet energy among all functions in $C^2(\Omega) \cap C\left(\bar{\Omega}\right)$ with boundary value h.*

Proof. Note that

$$\begin{aligned} E(w) &= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x}) - \nabla u(\mathbf{x}) + \nabla u(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \\ &= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x}) - \nabla u(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} + \int_\Omega \nabla(w(\mathbf{x}) - u(\mathbf{x}))\cdot\nabla u(\mathbf{x})\,\mathrm{d}\mathbf{x} + E(u). \end{aligned}$$

Suppose that u solves (5.1.3), then using that $\Delta u = 0$ and $u - w = 0$ on $\partial\Omega$ it follow by integration by parts that

$$\begin{aligned} &\int_\Omega \nabla(w(\mathbf{x}) - u(\mathbf{x}))\cdot\nabla u(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &\qquad = -\int_\Omega (w(\mathbf{x}) - u(\mathbf{x}))\Delta u(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega}(w(\sigma) - u(\sigma))\partial_{\mathbf{n}}u(\sigma)\,\mathrm{d}\sigma = 0. \end{aligned}$$

Consequently,

$$E(w) = \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x}) - \nabla u(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} + E(u) \geq E(u).$$

On the other hand, let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\bar{\Omega}\right)$ with $u = h$ on $\partial\Omega$. Suppose that u minimises $E(u)$. Let $w \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\bar{\Omega}\right)$ be such that $w = 0$. Now define $\phi : [-1, 1] \to [0, \infty)$ by

$$\phi(t) = E(u + tw) = E(u) + t\int_\Omega \nabla u(\mathbf{x}) \cdot \nabla w(\mathbf{x}) \,\mathrm{d}\mathbf{x} + t^2 E(w).$$

Note that $\phi'(0) = 0$. Therefore, as

$$\phi'(t) = \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla w(\mathbf{x}) \,\mathrm{d}\mathbf{x} + 2tE(w)$$

we deduce that

$$0 = \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla w(\mathbf{x}) \,\mathrm{d}\mathbf{x}.$$

for all $w \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\bar{\Omega}\right)$. Using integration by parts it follows that

$$0 = \int_\Omega \Delta u(\mathbf{x}) w(\mathbf{x}) \,\mathrm{d}\mathbf{x},$$

for all $w \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\bar{\Omega}\right)$ which implies that $\Delta u = 0$ in Ω. □

For the more general problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$$

the solution instead minimizes the Euler-Lagrange energy which is associated with the Dirichlet energy and is given by

$$E(w) = \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x} + \int_\Omega w(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}.$$

5.2 The Newtonian Potential

5.2.1 The Fundamental Solution

A solution to Laplace's equation can be characterized by some invariant properties.

1. Translational invariance. That is, if $u(\mathbf{x})$ is harmonic then so is $u(\mathbf{x} - \mathbf{y})$.
2. Rotational Invariance. That is, if $u(\mathbf{x})$ is harmonic and M is a rotation matrix, then $u(M\mathbf{x})$ is also harmonic.

Definition 5.2.1. *The function*

$$\Phi(\mathbf{x}) = \begin{cases} -\frac{1}{2\pi}\log(|\mathbf{x}|) & d = 2 \\ \frac{1}{(d-2)\omega_d}\frac{1}{|\mathbf{x}|^{d-2}} & d \geq 3 \end{cases}$$

is called the fundamental solution of the Laplace operator, Δ.

Exercise 5.2.2. *Using the invariant properties that a solution to Laplace's equation satisfies, verify the fundamental solution for the case when $d = 2$.*

Remark 5.2.3. *Note that $-\Delta\Phi(\mathbf{x} - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$. This arises due to the choice of constants made when constructing the fundamental solution.*

5.2.2 The Newtonian Potential in Three-Dimensions

Definition 5.2.4. *For a function f, the Newtonian potential of f is given by the convolution between f and the fundamental solution Φ.*

Proposition 5.2.5. *For $d = 3$ and a function f, the Newtonian potential of f, u, is given by*

$$u(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}. \tag{5.2.1}$$

Moreover, if f is smooth with compact support then

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}).$$

Theorem 5.2.6. *Let $f \in \mathcal{C}^2\left(\mathbb{R}^3\right)$ with compact support. Let u be the Newtonian potential of f given by (5.2.1). Then, u is the only solution in $\mathbb{R}^3$ of $-\Delta u = f$ such that $u \in \mathcal{C}^2\left(\mathbb{R}^3\right)$ vanishes at infinity.*

5.2.3 Application to div and curl

We can use the Newtonian potential to solve two problems.

1. Given a scalar field f and a vector field ω, we want to find a vector field $\mathbf{u}$ such that
   (a) $\nabla \cdot \mathbf{u} = f$, and
   (b) $\nabla \times \mathbf{u} = \omega$.
2. Given a three-dimensional vector field $\mathbf{u}$, we want to find φ and a vector field ψ such that
   (a) $\mathbf{u} = \nabla\varphi + \nabla \times \psi$, which is known as the Helmholtz decomposition.

Theorem 5.2.7. *Let $f \in \mathcal{C}^2\left(\mathbb{R}^3\right)$ and $\omega \in C^3\left(\mathbb{R}^3\right)$ with compact support and $\nabla \cdot \omega = 0$. Then the unique solution to Problem 1. that vanishes at infinity is given by*

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi}\nabla\left(\int_{\mathbb{R}^3} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}\right) + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \omega(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}. \tag{5.2.2}$$

Proof. Suppose that $\mathbf{u}_1$ and $\mathbf{u}_2$ are such solutions and let $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$. It follows that

$$\begin{cases} \nabla \cdot \omega = 0 \\ \nabla \times \omega = 0. \end{cases}$$

From $\nabla \times \mathbf{w} = 0$ it follows that $\nabla U = \omega$ for some scalar function U. Therefore,

$$\Delta U = \nabla \cdot (\nabla U) = \nabla \cdot \omega = 0$$

which implies that U is harmonic. Consequently, its derivatives, which are the components v_j of ω, are bounded harmonic functions in $\mathbb{R}^3$. Applying Corollary 5.1.12 it follows that each v_j is constant and so identically zero. Therefore, a solution to Problem 1. that vanishes at infinity is unique. To show existence we split $\mathbf{u} = \mathbf{v} + \mathbf{z}$ such that

$$\begin{cases} \nabla \cdot \mathbf{z} = 0, \\ \nabla \times \mathbf{z} = \omega \end{cases}$$

and

$$\begin{cases} \nabla \cdot \mathbf{v} = f, \\ \nabla \times \mathbf{v} = \mathbf{0}. \end{cases}$$

As $\nabla \times \mathbf{v} = \mathbf{0}$ there is some scalar field ϕ such that $\nabla\phi = \mathbf{v}$. Therefore, using $\nabla \cdot \mathbf{v} = f$ we have that $\Delta\phi = f$. Therefore, ϕ is given by the Newtonian potential of f, that is

$$\phi(\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}$$

allowing us to find $\mathbf{v}$. Now recall the identity

$$\nabla \times \nabla \times \mathbf{z} = \nabla(\nabla \cdot \mathbf{z}) - \Delta\mathbf{z}.$$

So as $\nabla \cdot \mathbf{z} = 0$ we have

$$-\Delta\mathbf{z} = \nabla \times \nabla \times \mathbf{z} = \nabla \times \omega.$$

Therefore,

$$\mathbf{z}(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \omega(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}.$$

□

Corollary 5.2.8. *The solution to Problem 2. is given by*

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \cdot \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y},$$

and

$$\psi(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y}.$$

Proof. Using Theorem 5.2.7 on $\mathbf{u}$, $\nabla \cdot \mathbf{u}$ and $\nabla \times \mathbf{u}$ we have that

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi}\nabla\int_{\mathbb{R}^3} \frac{\nabla \cdot \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \nabla \times \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}.$$

As $\mathbf{u}$ is vanishing at infinite we have

$$\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \nabla \times \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} = \frac{1}{4\pi}\nabla \times \int_{\mathbb{R}^3} \frac{\nabla \times \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}.$$

Therefore,

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \cdot \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}$$

and

$$\psi(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}.$$

□

5.2.4 Application to Fluid Dynamics

Consider the three-dimensional flow of a constant unit density fluid under the following assumptions.

1. The fluid has viscosity $\nu > 0$.
2. The conservative external force is given by $\mathbf{F} = \nabla f$.
3. $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is its velocity field.
4. The hydrostatic pressure is given by $p = p(t, \mathbf{x})$.

The Naiver-Stokes equations

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \nabla f \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

arise as a consequence of the conservation of mass and linear momentum applied to these assumptions. The non-linearity in the Naiver-Stokes equations makes the problem challenging to solve. However, under certain physical conditions, such as slow-flowing liquid, the contribution $(\mathbf{u} \cdot \nabla)\mathbf{u}$ can be assumed to be negligible. In such cases, we can also consider imposing boundary conditions and make progress in solving the problem. More specifically, we can consider solving the linear Naiver-Stokes problem

$$\begin{cases} \partial_t \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \nabla f \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u}(0, \mathbf{x}) = g(\mathbf{x}) \qquad\qquad \mathbf{x} \in \mathbb{R}^3, \end{cases} \tag{5.2.3}$$

where g is assumed to be divergence-free. To make progress on this problem we introduce vorticity, $\mathbf{w} = \nabla \times \mathbf{u}$. We can take the curl of the equations of (5.2.3) to arrive at

$$\begin{cases} \partial_t \mathbf{w} = v \nabla \mathbf{w} \\ \omega(0, \mathbf{x}) = \nabla \times g(\mathbf{x}). \end{cases} \tag{5.2.4}$$

Suppose further that the condition that $\mathbf{g} \in \mathcal{C}^2\left(\mathbb{R}^3\right)$ and $\nabla \times \mathbf{g}$ is bounded with compact support. Then one finds that (5.2.4) is a global Cauchy problem for the heat equation and so

$$\omega = \frac{1}{(4\pi\nu t)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \exp\left(-\frac{|\mathbf{x} - \mathbf{y}|^2}{4\nu t}\right) \nabla \times \mathbf{g}(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

Hence, we can use (5.2.2) to determine $\mathbf{u}$. Returning to (5.2.3), the pressure p satisfies

$$\nabla p = -\partial_t \mathbf{u} + \nu \Delta \mathbf{u} + \nabla f = \nu \Delta(\mathbf{u} - \mathbf{w}) + \nabla f.$$

The curl of the right-hand side is zero and so we can solve for p up to an additive constant.

5.3 Solution to Exercises

Exercise 5.2.2

Solution. We can write the Laplace operator in polar coordinates as

$$\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}.$$

As we require rotational invariance we can focus on $u = u(r)$ and look to solve

$$\partial_{rr} u + \frac{1}{r}\partial_r u = 0.$$

Using the substitution $r = e^z$ we note that

1. $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial z}$, and
2. $\frac{\partial^2 u}{\partial r^2} = \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial z}\right)$.

Which allows us to simplify our problem to

$$\frac{\partial^2 u}{\partial z^2} = 0$$

to give a solution of $u = c_0 + c_1 z = c_0 + c_1 \log(r)$. Returning to Cartesian coordinates we get the expected result. ☐