Jordan Algebras

MATH50002 Group Research Project

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Abstract

As part of the MATH50002 Group Research Project, our group conducted a survey into Jordan algebras, with the aim of developing a self-contained introduction to the algebraic structure devised by P. Jordan in the 1930s. The foundational work regarding Jordan algebras was written as a series of articles in the 1930s and 1940s. The main theorems are scattered and have proofs that are dependent on the material from other articles. Our group has gone through the literature to collect some of the main results into a coherent document and reconstructed their proofs from the various articles. A lot of the work of the project has been in updating notation and terminology to be consistent with modern mathematics. The report works up to the classification result for Jordan algebras over algebraically closed fields. Along the way, general results for Jordan algebras are given, solvable and nilpotent Jordan algebras are explored, Jordan algebras are investigated using the trace form, and simple Jordan algebras are discussed. The report also includes an introduction to general algebras, discusses the Wedderburn-Artin theorem, and the connection from Jordan algebras to the more well-known Lie algebras, through triple systems.

Introduction

Jordan algebras were proposed by P. Jordan in the early 1930s with the intention to encapsulate the algebraic properties of quantum mechanical systems. Measurements from quantum mechanical systems are represented by Hermitian matrices, therefore, Jordan wanted an algebraic structure on which operations would preserve this property. After investigating operations on Hermitian matrices Jordan gave a set of axioms from which he believed his desired structure would emerge. Work then started to explore these axioms and deduce their manifestations, with the ultimate goal of finding an infinite-dimensional algebra under this structure that would accommodate quantum mechanics. During the 1930s to the 1940s much of the theory for so-called Jordan algebras was developed. Jordan, von Neumann, and Wigner wrote the first article which defined the axioms and then A. Albert extended their work by proving some fundamental properties of these Jordan algebras. A. Albert supervised other mathematicians in this new field including Kalisch who contributed many more foundational results which lead to A.Albert giving a classification of finite-dimensional Jordan algebras in the 1940s. Throughout the 1950s and 1960s, these results were developed with the theory starting to venture into the infinite-dimensional setting. However, the results were not hinting at the existence of an infinite-dimensional Jordan algebra with the required properties to successfully describe quantum mechanics. In the late 1970s, Zelmanov provided a classification of infinite-dimensional Jordan algebras which conclusively showed that Jordan's proposed framework would not be able to encompass quantum mechanics. This was particularly disappointing for physicists, however, the study of Jordan algebras is definitely not futile. For example, the automorphism group of a particular Jordan algebra manifests as the F_4 exceptional Lie group. Furthermore, Zelmanov used the theory of Jordan algebras to solve the restricted Burnside problem. Therefore, despite not being adequate for quantum mechanics, Jordan algebras pose a rich theory that can be applied to various other problems.

In section 1 and section 2 of this report, general notions of algebras are introduced. In section 3, Jordan algebras are introduced and their basic properties are given which leads to more sophisticated results in sections 4,5,6, and 7. Section 4 discusses solvable and nilpotent Jordan algebras, the decomposition of semisimple algebras is explored in section 5, and section 6 looks at simple Jordan algebras. In section 7 the specific structure theorems are given, which will introduce the exceptional Jordan algebra. The exceptional Jordan algebras were of interest to physicists looking to use Jordan algebras to describe their quantum mechanical systems. It was A. Albert who first showed that only one finite-dimensional simple exceptional formally real Jordan algebra existed, which is now called Albert's algebra. It was Albert's algebra whose automorphism group was shown to be isomorphic to the exceptional Lie group F_4. Zelmanov went on to prove that Albert's algebra was the only simple exceptional Jordan algebra and no new infinite-dimensional exceptional Jordan algebra emerged. Finally, in section 8, we will investigate how Jordan algebras relate to the more well-known Lie algebras.

1 Introduction to Algebras

1.1 Definitions

In this first section, we give some definitions regarding more general algebras.

Definition 1.1. *An algebra is a vector space $\mathcal{A}$ over a field F with a bilinear function*

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (x, y) \mapsto xy.$$

This bilinear function is called the multiplication of the algebra.

Definition 1.2. *An algebra $\mathcal{A}$ is called unital if it contains an element $e \in A$ with the property that*

$$ex = xe = x \quad \forall x \in \mathcal{A}.$$

Definition 1.3. *An algebra $\mathcal{A}$ is associative if*

$$(xy)z = x(yz) \quad \forall x, y, z \in \mathcal{A}.$$

Definition 1.4. *An algebra $\mathcal{A}$ is commutative if*

$$xy = yx \quad \forall x, y \in \mathcal{A},$$

and anti-commutative if

$$xy = -yx \quad \forall x, y \in \mathcal{A}.$$

Definition 1.5. *An involution of an algebra $\mathcal{A}$ is a linear map $J : \mathcal{A} \to \mathcal{A}$ satisfying the properties*

$$J(J(x)) = x \text{ and } J(xy) = J(y)J(x) \quad \forall x, y, \in \mathcal{A}.$$

Definition 1.6. *For an associative algebra $\mathcal{A}$ define the center, $Z(\mathcal{A})$, as the set of all elements in $\mathcal{A}$ that commute with every other element of $\mathcal{A}$. In other words,*

$$Z(\mathcal{A}) := \{x \in \mathcal{A} : xy = yx, \ \forall y \in \mathcal{A}\}.$$

Definition 1.7. *For an associative algebra $\mathcal{A}$ define the centralizer of an element as the set of elements that commute with that element. That is, for $a \in \mathcal{A}$ the centralizer of a, $C(a)$, is the set*

$$C(a) := \{x \in \mathcal{A} : xa = ax\}.$$

Definition 1.8. *The derived series of an algebra $\mathcal{A}$ over a field F is the sequence*

$$\mathcal{A}^{(1)} = \mathcal{A}, \quad \mathcal{A}^{(n+1)} = \left[\mathcal{A}^{(n)}, \mathcal{A}^{(n)}\right] := \mathrm{Sp}\left(\left\{xy : x, y \in \mathcal{A}^{(n)}\right\}\right).$$

Definition 1.9. *The central series of an algebra $\mathcal{A}$ over a field F is the sequence*

$$\mathcal{A}^{1} = \mathcal{A}, \quad \mathcal{A}^{n+1} = [\mathcal{A}, \mathcal{A}^{n}] := \mathrm{Sp}\left(\{xy : x \in \mathcal{A}, y \in \mathcal{A}^{n}\}\right).$$

Definition 1.10. *An algebra $\mathcal{A}$ is solvable (nilpotent) if $\mathcal{A}^{(m)} = 0$ ($\mathcal{A}^{m} = 0$) for some positive integer m. Similarly, an element $x \in \mathcal{A}$ is nilpotent if there exists a positive integer m such that $x^m = 0$.*

1.2 Example

Let $\mathcal{A}$ be a unital algebra over a field F, with the involution $x \mapsto J(x)$ such that

$$x + J(x),\ xJ(x) \in F \quad \forall x \in \mathcal{A}.$$

Define the algebra $(\mathcal{A}, J)$ as the vector space $\mathcal{A} \oplus \mathcal{A}$, with the operations

- $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$,
- $c(x_1, x_2) = (cx_1, cx_2)$, and
- $(x_1, x_2)(y_1, y_2) = (x_1 y_1 - y_2 J(x_2), J(x_1) y_2 + y_1 x_2)$

for $x_i, y_i \in \mathcal{A}$ and $c \in F$. The involution J can be extended to $(\mathcal{A}, J)$ by defining

$$J(x_1, x_2) = (J(x_1), -x_2).$$

This process of generating an algebra from an existing algebra is known as the *Cayley-Dickson doubling process.* Explicitly, if one takes F=$\mathbb{R}$, and the involution to just be the identity map, one constructs an algebra isomorphic to $\mathbb{C}$. Continuing this process we construct the quaternions ($\mathbb{H}$) and the octonions ($\mathbb{O}$). These algebras and other algebras constructed in this way will be central to the classification theorems of Jordan algebras.

1.3 Ideals

Definition 1.11. *A subset $\mathcal{I}$ of an algebra $\mathcal{A}$ is called a left ideal if $\mathcal{A}\mathcal{I} := \mathrm{Sp}(\{ax : a \in \mathcal{A}, x \in \mathcal{I}\}) \subseteq \mathcal{I}$. Similarly, $\mathcal{I}$ is a right ideal if $\mathcal{I}\mathcal{A} \subseteq \mathcal{I}$. If an ideal is both a left and a right ideal then $\mathcal{I}$ is called a two-sided ideal.*

Remark 1.12. *When $\mathcal{A}$ is commutative, any left/right ideal is a two-sided ideal. In these cases one often omits the prefix and simply refers to such sets as ideals.*

Definition 1.13. *An algebra $\mathcal{A}$ is said to be simple if it has no ideals apart from $\{0\}$ and $\mathcal{A}$.*

Definition 1.14. *An ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ is said to be minimal if for any other ideal $\mathcal{J}$ of $\mathcal{A}$ such that $\mathcal{J} \subseteq \mathcal{I}$ then either $\mathcal{J} = \mathcal{I}$ or $\mathcal{J} = 0$.*

Lemma 1.15. *If $\mathcal{I}$ and $\mathcal{J}$ are ideals in $\mathcal{A}$, with $\mathcal{J}$ being minimal, then $\mathcal{J} \subseteq \mathcal{I}$ or $\mathcal{I} \cap \mathcal{J} = \{0\}$.*

Proof. As $\mathcal{J} \cap \mathcal{I}$ is an ideal of $\mathcal{J}$ and because $\mathcal{J}$ is minimal then either $\mathcal{J} \cap \mathcal{I} = \mathcal{J}$ or $\mathcal{J} \cap \mathcal{I} = \{0\}$. When $\mathcal{J} \cap \mathcal{I} = \mathcal{J}$ it follows that $\mathcal{J} \subseteq \mathcal{I}$. □

Definition 1.16. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two algebras over a field F. Then a linear transformation of vector spaces $\phi : \mathcal{A} \to \mathcal{A}'$ is called a homomorphism of algebras if*

$$\phi(xy) = \phi(x)\phi(y) \quad \forall x, y \in \mathcal{A}.$$

Theorem 1.17. *Let* $f : \mathcal{A} \to \mathcal{A}'$ *be a homomorphism between the algebras* $\mathcal{A}$ *and* $\mathcal{A}'$*. Then the kernel of* f*, defined as* $\ker(f) := \{x \in \mathcal{A} : f(x) = 0\}$*, is an ideal of* $\mathcal{A}$*.*

Proof. Firstly, let $x, y \in \ker(f)$. Then $f(x) = f(y) = 0$ and so by the linearity of the homomorphism

$$f(x + y) = f(x) + f(y) = 0 + 0 = 0.$$

Therefore, $x + y \in \ker(f)$. Similarly,

$$f(-x) = -f(x) = -0 = 0$$

so that $-x \in \ker(f)$. Hence, $\ker(f)$ forms a group under addition. Next let $a \in \mathcal{A}$ and $x \in \ker(f)$. Then

$$f(ax) = f(a)f(x) = f(a)0 = 0,$$

which implies that $ax \in \ker(f)$. This shows that $\ker(f)$ is an ideal of $\mathcal{A}$. $\square$

1.4 References

Material for the definition and example came from [14], [4] and [7]. Material regarding ideals came from [12].

2 Wedderburn-Artin Theorem

The Wedderburn-Artin theorem is a classification of finite-dimensional associative algebras. Later on, we will see that Jordan algebras are in general not associative so this theorem cannot be applied. However, this section will indicate some of the theory that may be necessary to develop to achieve similar results for Jordan algebras. When Jordan algebras were first introduced, mathematicians conducted investigations surrounding ideas similar to those outlined in this section to try and derive a similar classification result for Jordan algebras. In section 7 of this report we will see that indeed classification results for Jordan algebras were achieved, however, they are not as concise as the Wedderburn-Artin theorem.

2.1 Definitions

Definition 2.1. *Let $(R, +, \times)$ be a ring, and $(M, +)$ an additive abelian group. M is called a left-R module if there exists an R-action $(- \cdot -) : R \times M \to M$, satisfying:*

1. $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$,
2. $(r_1 \times r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$,
3. $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$, *and*
4. $1_R \cdot m = m$.

The concept of a module over a ring can be regarded as a generalization of vector spaces over a field. It is necessary to point out that every ring R is a left R-module itself by defining the left regular action:

$$L : R \times R \to R,\ (\alpha, x) \mapsto \alpha x.$$

Definition 2.2. *Suppose M is a left R-module and $S \subseteq M$. Then S is called a submodule of M if $(S, +)$ is a subgroup of $(M, +)$ and for all r in R, and s in S we have that $r \cdot s \in S$.*

Remark 2.3. *Every ring R is a left R-module, therefore, if S is a submodule of R it is also a left ideal of R.*

Definition 2.4. *Suppose M and N are two left R-modules, then the map $\phi : M \to N$ is called an R-module homomorphism (or an R-linear map) between M and N if for any $r \in R$ and $x, y \in M$ the following hold:*

1. $\phi(x + y) = \phi(x) + \phi(y)$, *and*
2. $\phi(r \cdot x) = r \cdot \phi(x)$.

Definition 2.5. *If ϕ is a bijective homomorphism between the left R-modules M and N, then ϕ is called an R-module isomorphism. If an isomorphism between M and N exists then they are said to be isomorphic, which we denote $M \cong_R N$.*

For R-modules M and N let $\text{Hom}_R(M,N)$ denote the set of all R-module homomorphisms between M and N. If M equals N then we use $\text{End}_R(M)$ to denote $\text{Hom}_R(M,M)$. Elements of $\text{End}_R(M)$ are called endomorphisms.

Definition 2.6. *Suppose $(R,+,\times)$ is a ring, the opposite of R is the ring $(R^{\text{op}},+,*)$ where $a*b = b\times a$.*

Definition 2.7. *For a ring R let S be the set of all $n\times n$ matrices with entries in R. Then $M_n(R) = (S,+,\cdot)$ is called a full matrix ring on R, where $+$ is matrix addition and $\cdot$ is matrix multiplication.*

Definition 2.8. *A R-module M is said to be simple if it's non-zero and it has no other submodules except for 0 and itself.*

Definition 2.9. *A module M is called semisimple if it is a direct sum of simple modules. An algebra $\mathcal{A}$ is called semisimple if all non-zero $\mathcal{A}$-modules are semisimple.*

2.2 Preliminary Results

Proposition 2.10. *Let R be a ring then:*

1. *$(R^{\text{op}})^{\text{op}} = R$, and*
2. *if R is commutative then $R^{\text{op}} = R$.*

Proof. Suppose the multiplication for $(R^{\text{op}})^{\text{op}}$ is $\otimes$ then

$$a\otimes b = b*a = a\times b$$

which implies that $\otimes = \times$. This means that $(R^{\text{op}})^{\text{op}} = R$.
If R is commutative then in R^{op} we have that

$$a*b = b\times a = a\times b$$

and therefore $R^{\text{op}} = R$. $\square$

Lemma 2.11. *Let M be a finitely generated R-module. The following properties are equivalent:*

1. *Any submodule of M is a direct summand. In other words, if $W\subset M$ is a submodule then there exists a submodule W' such that $M = W\oplus W'$.*
2. *M is semisimple.*
3. *M is a finite sum of simple submodules.*

Proof. Note that 1 implies 2 and 2 implies 3. For 3 implies 1 consider the submodules of M whose intersection with W is $\{0\}$. Let L be such a submodule of maximal dimension. If $W\oplus L\neq M$ then there is some simple submodule S in M that is not in $W\oplus L$, by the assumption that M is a finite sum of simple submodules. However, this would imply that the intersection of $S+L$ and W is also $\{0\}$ which contradicts the properties of L. $\square$

Proposition 2.12. *If a module M is semisimple, then so is every submodule and every quotient of M.*

Proof. If N is a submodule then $M = N \oplus L$ for some L by Lemma 2.11. This implies $M/L \cong N$, and so it is enough to prove the result for quotient modules. If M/L is a quotient module consider the projection homomorphism π from M to M/L. Write M as a sum of simple modules S_i and then $\pi(S_i)$ is either simple or $\{0\}$. This proves that M/L is a sum of simple modules, and so the result follows from Lemma 2.11. □

Proposition 2.13. *An algebra $\mathcal{A}$ is semisimple if and only if the $\mathcal{A}$-module is semisimple.*

Proof. Suppose $\mathcal{A}$ to be semisimple as a $\mathcal{A}$-module. Let M be an $\mathcal{A}$-module and choose a set $\{m_1, ..., m_r\}$ of generators for M. Let $\mathcal{A}^r$ be the direct sum of r copies of A. Note that $\mathcal{A}^r$ is also semisimple as if $\mathcal{A} = \bigoplus_j \mathcal{A}_j$ where $\mathcal{A}_j$ are simple $\mathcal{A}$-modules then $\mathcal{A}^r = \bigoplus_r \bigoplus_j \mathcal{A}_j$. Define the map $\phi : \mathcal{A}^r \to M$ by

$$(a_1, \ldots, a_r) \mapsto \sum_{i=1}^{r} a_i m_i.$$

Clearly, ϕ is a surjective homomorphism, thus according to the isomorphism theorem of modules, M is isomorphic to some quotient of a semisimple module $\mathcal{A}^r$. By Proposition 2.12 it follows that M is semisimple.
The converse of this proposition is trivial. □

Proposition 2.14. *Suppose that $\mathcal{A}$ is a semisimple algebra and let $\mathcal{A}_i$ be the collection of simple distinct $\mathcal{A}$-submodules of $\mathcal{A}$. Let M be an $\mathcal{A}$-module (which is automatically semisimple). Then there is a set of integers n_i such that*

$$M \cong \mathcal{A}_1^{n_1} \oplus \cdots \oplus \mathcal{A}_r^{n_r}.$$

Proof. It follows naturally from Lemma 2.11 and 2.13. □

Lemma 2.15. *Suppose $\mathcal{A}$ is a simple unital associative algebra over a field F, and let $\mathcal{I}, \mathcal{J}$ be two minimal left ideals of $\mathcal{A}$. Then there exists $\alpha \in \mathcal{A}$ such that $\mathcal{I} = \mathcal{J}\alpha = \{j\alpha : j \in \mathcal{J}\}$, thus $\mathcal{I} \cong \mathcal{J}$.*

Proof. Note that $\mathcal{IA}$ and $\mathcal{JA}$ are two-sided ideals of $\mathcal{A}$. As $\mathcal{A}$ simple, if $\mathcal{I}$ and $\mathcal{J}$ are both not trivial then

$$\mathcal{IA} = \mathcal{JA} = \mathcal{A}$$

which implies that

$$\mathcal{IJA} = \mathcal{IA} = \mathcal{A}$$

where $\mathcal{I}$, $\mathcal{J}$ are not zero. Let $\alpha \in \mathcal{J}$, then $\mathcal{I}\alpha \subseteq \mathcal{J}$ (by closure of left multiplication). Clearly $\mathcal{I}\alpha$ is a left ideal of $\mathcal{A}$, and since $\mathcal{J}$ is a minimal left ideal it follows that $\mathcal{I}\alpha = \mathcal{J}$. Hence, $\mathcal{I} \cong \mathcal{J}$. □

Lemma 2.16. *Suppose M_i and N_i for $i = 1, 2, ..., n$ are left R-modules, then:*

$$\mathrm{Hom}_R\left(\bigoplus_{i=1}^{n} M_i, N\right) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_R(M_i, N), \text{ and}$$
$$\mathrm{Hom}_R\left(M, \bigoplus_{i=1}^{n} N_i\right) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_R(M, N_i).$$

Proof. Using induction, it's sufficient to show

$$\mathrm{Hom}_R(M_1 \oplus M_2, N) \cong \mathrm{Hom}_R(M_1, N) \oplus \mathrm{Hom}_R(M_2, N).$$

Define a natural mapping $\Phi : \mathrm{Hom}_R(M_1 \oplus M_2, N) \to \mathrm{Hom}_R(M_1, N) \oplus \mathrm{Hom}_R(M_2, N)$ by

$$\phi(m_1, m_2) \mapsto \phi(m_1) \oplus \phi(m_2).$$

It's easy to verify that Φ is indeed an isomorphic mapping. □

Definition 2.17. *A division algebra $\mathcal{D}$ is a ring for which every non-zero element has a multiplicative inverse.*

Lemma 2.18 (Schur's lemma)**.** *Suppose M and N are two simple modules over a ring R, then any homomorphism $\phi : M \to N$ of R-modules is either an isomorphism or the zero map. In particular, the endomorphism ring of a simple module is a division ring.*

Proof. For all $\phi \in \mathrm{Hom}_R(M, N)$ note that $\ker(\phi) \leq M$ and $\mathrm{im}(\phi) \leq N$. By the simplicity of M and N, we can deduce that either

$$\ker(\phi) = M \text{ and } \mathrm{im}(\phi) = 0, \text{ or } \ker(\phi) = 0 \text{ and } \mathrm{im}(\phi) = N.$$

Therefore, ϕ is either the zero map or an isomorphism. Hence, $\mathrm{End}_R(M)$ is a division ring since its non-zero elements are invertible automorphisms. □

Lemma 2.19. *Suppose $\mathcal{A}$ is an algebra over the field F, then* $\mathrm{End}_{\mathcal{A}}(\mathcal{A}) \cong \mathcal{A}^{\mathrm{op}}$.

Proof. Define the right multiplication action of a fixed element α by

$$R_\alpha : A \to A, \quad x \mapsto x\alpha.$$

Next, define the mapping

$$R : \mathcal{A}^{\mathrm{op}} \to \mathrm{End}_A(A), \quad \alpha \mapsto R_\alpha.$$

First, to verify it's well-defined we show that R_α is indeed an endomorphism. To do this it is sufficient to note that

$$R_\alpha(x + y) = (x + y)\alpha = x\alpha + y\alpha = R_\alpha(x) + R_\alpha(y)$$

and

$$R_\alpha(rx) = rx\alpha = rR_\alpha(x), \forall x, y, r \in \mathcal{A}^{\mathrm{op}}.$$

Then as,

$$R(a \times b)(x) = R(b * a)(x) = xba = R(a)(xb) = R(a) \circ R(b)(x), \forall x \in \mathcal{A}$$

it follows that R is a multiplicative homomorphism. Next, to show that R is a bijection we note that for all $\phi \in \mathrm{End}_{\mathcal{A}}(\mathcal{A})$ and $x \in \mathcal{A}$ we have that

$$\phi(x) = \phi(1 \cdot x) = x\phi(1) = R(\phi(1))(x).$$

Since $\phi(1) \in \mathcal{A}$ we have that R is surjective. Furthermore, $\ker(R) = 0$ implies that R is injective, thus $\mathrm{End}_{\mathcal{A}}(\mathcal{A}) \cong \mathcal{A}^{\mathrm{op}}$. □

Lemma 2.20. *Suppose $M_n(\mathcal{D})$ is the matrix algebra over a division ring $\mathcal{D}$, then $M_n(\mathcal{D}) \cong (M_n(\mathcal{D}))^{\mathrm{op}}$.*

Proof. Define the mapping that transposes a matrix as

$$T : M_n(\mathcal{D}) \to (M_n(\mathcal{D}))^{\mathrm{op}}$$

$$A \mapsto A^{\top}, \quad a_{ij} \mapsto a_{ji}.$$

Note that

- $(A + B)^{\top} = A^{\top} + B^{\top}$,
- $(kA)^{\top} = kA^{\top}$,
- $(A \cdot B)^{\top} = B^{\top} \cdot A^{\top}$, and
- T is invertible.

The above shows that T is an isomorphism, so we conclude that $M_n(\mathcal{D}) \cong (M_n(\mathcal{D}))^{\mathrm{op}}$. □

Theorem 2.21. *Let $M_n(R)$ be a full matrix ring on the ring R, then any ideal $\mathcal{I}$ is of the form $M_n(\mathcal{I})$ for some ideal $\mathcal{I}$ of R.*

Proof. If $\mathcal{I}$ is an ideal of R, then as scalar multiplication and matrix addition happen component-wise it is clear that $M_n(\mathcal{I})$ is an ideal of $M_n(R)$. Furthermore, if $M_n(\mathcal{I}_1) = M_n(\mathcal{I}_2)$ for ideals $\mathcal{I}_1, \mathcal{I}_2$, it is clear that $\mathcal{I}_1 = \mathcal{I}_2$ because matrices are equal if and only if each component is equal.
Next, suppose that $\mathcal{J}$ is an ideal of $M_n(R)$. Let $\mathcal{I}$ denote the set of elements in the top left entry of the matrices of $\mathcal{J}$, then $\mathcal{I}$ is an ideal of R. This is because first, it's trivially closed under addition and secondly, if it's not closed under multiplication of elements in R, then it contradicts $\mathcal{J}$ is an ideal of $M_n(R)$. Let e_{ij} be the matrix with 0 in every entry apart from the ij^{th} entry. Let $M \in \mathcal{J}$, then $e_{1j}Me_{j1} = m_{ij}e_{11} \in \mathcal{J}$ so that $m_{ij} \in \mathcal{I}$ and hence $\mathcal{J} \subseteq M_n(\mathcal{I})$. On the other hand, let $N = (n_{ij}) \in M_n(\mathcal{I})$, and take $M = (m_{ij}) \in \mathcal{J}$ such that $m_{11} = n_{ij}$. Then $n_{ij}e_{ij} = m_{11}e_{ij} = e_{i1}Me_{1j} = m_{11}e_{ij} \in \mathcal{J}$. Therefore, as $\mathcal{J}$ is closed under addition, and ij were arbitrary it follows that $N \in \mathcal{J}$. Therefore, $M_n(\mathcal{I}) \subseteq \mathcal{I}$ which means that $M_n(\mathcal{I}) = \mathcal{J}$. □

Corollary 2.21.1. *If R is a simple ring, then $M_n(R)$ is also simple.*

Proof. By Theorem 2.21, all ideals of $M_n(R)$ are of the form $M_n(\mathcal{I})$ where $\mathcal{I}$ is an ideal of R. As R is simple it only has the ideals R and $\{0\}$, therefore, $M_n(R)$ only has the ideals $M_n(R)$ and $\{0\}$ meaning it is also simple. □

Definition 2.22. *A field F is said to be algebraically closed if every polynomial with coefficients in F has a root in F.*

Lemma 2.23. *Let $\mathcal{D}$ be a finite-dimensional division algebra over an algebraically closed field F. Then $\mathcal{D} = F$.*

Proof. For all $a \in \mathcal{D}$ and $a \neq 0$, as $\mathcal{D}$ is finite dimensional, $a, a^2, \dots a^k, \dots$ are linearly dependent. We then choose the smallest n where:

$$a^n + c_1 a^{n-1} + \cdots + c_n = 0.$$

Consider $f(x) = x^n + c_1 x^{n-1} + \cdots + c_n$, as F is algebraically closed, f has a root λ in F such that $f(x) = (x - \lambda)g(x)$ where $\deg(g(x)) = n - 1$. Inserting $x = a$ we get:

$$(a - \lambda)g(a) = 0,$$

and since $f(x)$ is chosen to be the smallest degree so $g(a) \neq 0$ hence invertible ($\mathcal{D}$ is division algebra). It follows that $a = \lambda \in F$, hence $\mathcal{D} = F$. □

2.3 Theorems

Theorem 2.24 (Wedderburn-Artin Theorem for Algebras)**.** *Let $\mathcal{A}$ be a unital associative finite-dimensional algebra over field F.*

1. *$\mathcal{A}$ is semisimple if and only if there exist division algebras $\mathcal{D}_1, \dots, \mathcal{D}_r$ over F with positive integers $n_1, \dots, n_r$ such that*

$$\mathcal{A} \cong M_{n_1}(\mathcal{D}_1) \oplus M_{n_2}(\mathcal{D}_2) \oplus \cdots \oplus M_{n_r}(\mathcal{D}_r).$$

2. *If F is an algebraically closed field, then $\mathcal{A}$ is semisimple if and only if there exist positive integers $n_1, \dots, n_r$, such that*

$$\mathcal{A} \cong M_{n_1}(F) \oplus M_{n_2}(F) \oplus \cdots \oplus M_{n_r}(F).$$

Remark 2.25. *Note that statement 2 can be derived easily from statement 1 since the only finite-dimensional associative division algebra on an algebraically closed field F is F itself by Lemma 2.23.*

Proof. (Theorem 2.24)
Suppose that $\mathcal{A}$ is semisimple and let $\mathcal{A}_i$ be the non-pairwise isomorphic simple submodules of $\mathcal{A}$. With $\mathcal{A} = \mathcal{A}_1^{n_1} \oplus \cdots \oplus \mathcal{A}_r^{n_r}$, we have that

$$\begin{aligned} \mathcal{A}^{\text{op}} \cong \text{End}_{\mathcal{A}}(\mathcal{A}) &\cong \text{End}_{\mathcal{A}}(\mathcal{A}_1^{n_1} \oplus \cdots \oplus \mathcal{A}_r^{n_r}) \\ &\cong \text{End}_{\mathcal{A}}(\mathcal{A}_1^{n_1}) \oplus \cdots \oplus \text{End}_{\mathcal{A}}(\mathcal{A}^{n_r}). \end{aligned}$$

Now we note that:

$$\begin{aligned}\mathrm{End}_{\mathcal{A}}(\mathcal{A}_i^{n_i}) = \mathrm{End}_{\mathcal{A}}\left(\bigoplus_{k=1}^{n_i}\mathcal{A}_i\right) &= \mathrm{Hom}_{\mathcal{A}}\left(\bigoplus_{k=1}^{n_i}\mathcal{A}_i, \bigoplus_{l=1}^{n_i}\mathcal{A}_i\right)\\ &\cong \bigoplus_{k=1}^{n_i}\bigoplus_{l=1}^{n_i}\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}_i, \mathcal{A}_i)\\ &= \bigoplus_{k=1}^{n_i}\bigoplus_{l=1}^{n_i}\mathrm{End}_{\mathcal{A}}(\mathcal{A}_i).\end{aligned}$$

By installing everything into the entries of a matrix, we have:

$$\bigoplus_{k=1}^{n_i}\bigoplus_{l=1}^{n_i}\mathrm{End}_{\mathcal{A}}(\mathcal{A}_i) \cong M_{n_i}(\mathrm{End}_{\mathcal{A}}(\mathcal{A}_i)).$$

By Schur's lemma, since $\mathcal{A}_i$ is simple module, $\mathrm{End}_{\mathcal{A}}(\mathcal{A}_i)$ is a division algebra. Letting $\mathcal{D}_i = \mathrm{End}_{\mathcal{A}}(\mathcal{A}_i)$, we have

$$\mathcal{A}^{\mathrm{op}} \cong M_{n_1}(\mathcal{D}_1) \oplus M_{n_2}(\mathcal{D}_2) \oplus \cdots \oplus M_{n_r}(\mathcal{D}_r).$$

Then take the opposite to deduce that

$$\mathcal{A} = (\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \cong M_{n_1}(\mathcal{D}_1) \oplus M_{n_2}(\mathcal{D}_2) \oplus \cdots \oplus M_{n_r}(\mathcal{D}_r).$$

□

Theorem 2.26 (Wedderburn-Artin Theorem for Rings). *If R is a unital ring, then R is semisimple if and only if there exist division rings $\mathcal{D}_1$, ..., $\mathcal{D}_r$ and positive integers $n_1, \ldots, n_r$, such that*

$$\mathcal{A} \cong M_{n_1}(\mathcal{D}_1) \oplus M_{n_2}(\mathcal{D}_2) \oplus \cdots \oplus M_{n_r}(\mathcal{D}_r).$$

Proof. The proof of this statement requires material not sufficiently developed in this report. To see the proof refer to [11]. □

2.4 References

Material for this section came from [17] and [15].

3 A General Theory of Jordan Algebras

In this section, the more specific exploration of Jordan algebras begins. The defining axioms for Jordan algebras were devised by Jordan after investigating Hermitian matrices. However, he intended to explore how this structure could be used more generally.

Definition 3.1. *An algebra $\mathcal{J}$ over a field F satisfying*

1. $x \circ y = y \circ x$ *(Commutativity) and,*
2. $x \circ \left(x^2 \circ y\right) = x^2 \circ (x \circ y)$ *(Jordan's Identity)*

is known as a Jordan algebra.

Definition 3.2. *The plus algebra $\mathcal{A}^+$ of an associative algebra $\mathcal{A}$ over a field F ($\mathrm{char}(F) \neq 2$) is the algebra with the same underlying vector space as $\mathcal{A}$ but has multiplication defined as*

$$x \circ y = \frac{1}{2}(xy + yx).$$

Theorem 3.3. *Let $\mathcal{A}$ be an associative algebra, then $\mathcal{A}^+$ is a Jordan algebra.*

Proof. It is clear that the product is commutative as

$$x \circ y = \frac{1}{2}(xy + yx) = \frac{1}{2}(yx + xy) = y \circ x.$$

Furthermore, it satisfies the Jordan identity as

$$\begin{aligned}
(x \circ y) \circ x^2 &= \frac{1}{2}\left((x \circ y)\, x^2 + x^2\, (x \circ y)\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\,(xy + yx)\, x^2 + \frac{1}{2} x^2\, (xy + yx)\right) \\
&= \frac{1}{2}\left(\frac{1}{2} xyx^2 + \frac{1}{2} yx^3 + \frac{1}{2} x^3 y + \frac{1}{2} x^2 yx\right) \\
&= \frac{1}{2}\left(x \frac{1}{2}\left(yx^2 + x^2 y\right) + \frac{1}{2}\left(yx^2 + x^2 y\right) x\right) \\
&= \frac{1}{2}\left(x\left(y \circ x^2\right) + \left(y \circ x^2\right) x\right) \\
&= x \circ \left(y \circ x^2\right).
\end{aligned}$$

□

Definition 3.4. *A special Jordan algebra is one which is isomorphic to a subalgebra of $\mathcal{A}^+$ for some associative algebra $\mathcal{A}$, otherwise, it is an exceptional Jordan algebra.*

Remark 3.5. *The operation $x \circ y = \frac{1}{2}(xy + yx)$ is often referred to as the Jordan product.*

Definition 3.6. *A Euclidean Jordan algebra is a Jordan algebra that is equipped with an bilinear product that is compatible with the Jordan product. In other words, the bilinear product satisfies*

$$\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$$

for all x, y, z in the algebra.

3.1 Jordan Algebra Identities

The following identities are fundamental to the properties of Jordan algebras as they manifest from the defining axioms. Some of the identities will play important roles in the proof of the main theorems of this report. Throughout let $\mathcal{J}$ be a Jordan algebra over an infinite field F with multiplication denoted by $\circ$.

Lemma 3.7. *(Polarization Formula) For a Jordan algebra $\mathcal{J}$ the following identity holds*

$$2(z \circ y) \circ (x \circ z) + z^2 \circ (y \circ z) = 2x \circ (y \circ (x \circ z)) + \left(x^2 \circ y\right) \circ z$$

for all $x, y, z \in \mathcal{J}$.

Proof. For $\lambda \in F$ and $z \in \mathcal{J}$ substitute $x = x + \lambda z$ into the Jordan Identity. Expanding the left hand side yields

$$\begin{aligned} &x \circ \left(x^2 \circ y\right) + \lambda x((x \circ z) \circ y) + \lambda x((z \circ x) \circ y) + \lambda z \circ \left(x^2 \circ y\right) \\ &+ \lambda^2 x \circ \left(z^2 \circ y\right) + \lambda^2 z \circ ((x \circ z) \circ y) + \lambda^2 z \circ ((z \circ x) \circ y) + \lambda^3 z \circ \left(z^2 \circ y\right) \end{aligned}$$

and expanding the right hand side yields

$$\begin{aligned} &x^2 \circ (x \circ y) + \lambda(x \circ z) \circ (x \circ y) + \lambda(z \circ y) \circ (x \circ y) + \lambda x^2 \circ (z \circ y) \\ &+ \lambda^2 z^2 \circ (x \circ y) + \lambda^2 (x \circ z) \circ (z \circ y) + \lambda^2 (z \circ y) \circ (z \circ y) + \lambda^3 z^2 \circ (z \circ y). \end{aligned}$$

By comparing λ coefficients and using the commutativity of $\circ$ the result follows. □

Definition 3.8. *For a Jordan algebra $\mathcal{J}$ and $x \in \mathcal{J}$, define $R(x)$ to be the linear transformation representing right multiplication by x. So that for $y \in \mathcal{J}$ we have*

$$R(x)y = y \circ x.$$

Similarly, define $L(x)$ to be the linear transformation representing left multiplication by x. So that for $y \in \mathcal{J}$ we have

$$L(x)y = x \circ y.$$

Remark 3.9. *Note that a Jordan algebra is commutative so that $R(x) = L(x)$ for any $x \in \mathcal{J}$.*

Remark 3.10. *Considering the result of Lemma 3.7 as a linear transformation in z gives the identity*

$$2R(x \circ y)R(x) + R\left(x^2\right)R(y) = 2R(x)R(y)R(x) + R\left(x^2 \circ y\right).$$

Through a similar method of proof to that of Lemma 3.7 but applied instead to the polarization formula yields the identity

$$R(x \circ y)R(z) + R(x \circ z)R(y) + R(y \circ z)R(x) = \\ R(z)R(x \circ y) + R(y)R(x \circ z) + R(x)R(y \circ z). \quad (1)$$

As the above identity is a linear transformation we can apply it to an element w. Then interchanging the z and the w results in the identity

$$R((x \circ y) \circ z) = R(x)R(y \circ z) + R(y)R(x \circ z) \\ + R(z)R(x \circ y) - (R(x)R(z)R(y) + R(y)R(z)R(x)). \quad (2)$$

Again by using (1) it follows that

$$R((x \circ y) \circ z) = R(y \circ z)R(x) + R(x \circ z)R(y) \\ + R(x \circ y)R(z) - (R(x)R(z)R(y) + R(y)R(z)R(x)). \quad (3)$$

By interchanging x and z in (2) then subtracting (2) by the result gives

$$R((x \circ y) \circ z) - R(x \circ (y \circ z)) = (R(z)R(x) - R(x)R(z))R(y) \\ - R(y)(R(z)R(x) - R(x)R(z)). \quad (4)$$

Next, if one defines powers inductively as $x^{t+1} = x^t \circ x$, then using (3) it follows that

$$R\left(x^{t+1}\right) = 2R\left(x^t\right)R(x) + R\left(x^2\right)R\left(x^{t-1}\right) \\ - \left(R\left(x^{t-1}\right)R\left(x^2\right) + R(x)^2R\left(x^{t-1}\right)\right). \quad (5)$$

Remark 3.11. *From* (5) *it follows by induction that for any $x \in \mathcal{J}$ and positive integer t we have that $R\left(x^t\right)$ is a polynomial in $R(x)$ and $R\left(x^2\right)$.*

3.2 Basic Results

We now give some basic properties of Jordan algebras. Some may not contribute to the proof of the main theorems of this report, however, they do illustrate some interesting properties of Jordan algebras. Throughout let $\mathcal{A}$ be an associative algebra.

Proposition 3.12. *Powers of $x \in \mathcal{A}^+$ coincide with the powers of $x \in \mathcal{A}$.*

Proof. Let x^n denote the n^{th} power of x in $\mathcal{A}$, and let $x^{(n)}$ denote the n^{th} power of x in $\mathcal{A}^+$. It is clear that $x^1 = x^{(1)}$. We now proceed by induction. Assume that for $k \leq n$ we have that $x^k = x^{(k)}$. Then,

$$x^{(n+1)} = x \circ x^{(n)} = x \circ x^n = \frac{1}{2}(xx^n + x^n x) = x^{n+1}.$$

□

Proposition 3.13. *For $x, y \in \mathcal{A}$ the following are equivalent:*

1. *x and y are commuting in $\mathcal{A}^+$, i.e. $x \circ (y \circ z) = y \circ (x \circ z)$ for all $z \in \mathcal{A}$, and*

2. *$xy - yx$ is in the center of $\mathcal{A}$.*

Proof. If $x \circ (y \circ z) = y \circ (x \circ z)$ then expanding the Jordan products it must be the case that

$$\frac{1}{2}\left(x\frac{1}{2}(yz + zy) + \frac{1}{2}(yz + zy)x\right) = \frac{1}{2}\left(y\frac{1}{2}(xz + zx) + \frac{1}{2}(xz + zx)y\right)$$

from which it follows that

$$(xy - yx)z = z(xy - yx).$$

□

Proposition 3.14. *Suppose $\mathcal{A}$ has unit element e, then e is also a unit element of $\mathcal{A}^+$.*

Proof. For $x \in \mathcal{A}^+$ it follows that

$$x \circ e = \frac{1}{2}(xe + ex) = \frac{1}{2}(2x) = x.$$

By commutativity, we also have $e \circ x = x$. Therefore, e is a unit element of $\mathcal{A}^+$. □

Theorem 3.15. *For a Jordan algebra $\mathcal{J}$, over a field F, and $x \in \mathcal{J}$ it follows that for any positive integers s and t,*

$$x^s \circ x^t = x^{s+t}.$$

Proof. Consider $\mathcal{J}_0$ as the associative algebra generated by $R(x)$ and $R\left(x^2\right)$ (that is the span of finite sums and products of $R(x)$ and $R\left(x^2\right)$). Then by Remark 3.11 $\mathcal{J}_0$ contains $R\left(x^t\right)$ for all positive integers t. In particular, this implies that $R\left(x^t\right) R\left(x^s\right) = R\left(x^s\right) R\left(x^t\right)$ for all positive integers s and t. Using this we see that for $t > 1$ the following holds

$$x^s \circ x^t = x^s \left(x^{t-1} \circ x\right) = R\left(x^s\right) R\left(x^{t-1}\right) x = R\left(x^{t-1}\right) R\left(x^s\right) x = x^{s+1} x^{t-1}.$$

Therefore,

$$x^s \circ x^t = x^{s+1} \circ x^{t-1} = \cdots = x^{s+t-1} \circ x = x^{s+t}.$$

□

Remark 3.16. *Theorem 3.15 says that Jordan algebras are power associative.*

Lemma 3.17. *Let $\mathcal{J}$ be a Jordan algebra and let $x, y \in \mathcal{J}$. Then x and $x \circ y$ commute if and only if x^2 and y commute.*

Proof. Letting $z = x$ in (1) gives

$$2R(x \circ y)R(x) + R\left(x^2\right)R(y) = 2R(x)R(x \circ y) + R(y)R\left(x^2\right)$$

which rearranges to give

$$2R(x \circ y)R(x) = 2R(x)R(x \circ y) = R(y)R\left(x^2\right) - R\left(x^2\right)R(y).$$

Therefore, the transformations $R(x)$ and $R(x \circ y)$ commute if and only if the transformations $R\left(x^2\right)$ and $R(y)$ commute. □

Theorem 3.18. *For a Jordan algebra $\mathcal{J}$ defined over a field F with $x, y \in \mathcal{J}$, then the following assertions are equivalent:*

1. *x, y and $x \circ y$ commute pairwise,*
2. *$F[x, y]$ (the algebra of polynomials in variables x, y with coefficients in F) is an associative subalgebra of $\mathcal{J}$ whose elements commute pairwise.*

Proof. Since x, y and $x \circ y$ belong to $F[x, y]$, it is clear that 2. implies 1. The proof of the implication for the other direction is done in 3 steps.
Step 1: All the terms x^r, y^s, $x \circ y$ commute by pairs and we have

$$y^q \circ (x^r \circ y^s) = x^r \circ y^{q+s},\ x^p \circ (x^r \circ y^s) = x^{p+r} \circ y^s.$$

From Lemma 3.17 we have that x^2, y commute and similarly y^2, x commute. By Remark 3.10 we also see that $x^2 \circ y, x, y$ commute by pairs. Again from applying Lemma 3.17 to y and x^2 we have that x^2, y^2 commute. Then applying it again to x and $x \circ y$ we have that $x^2, x \circ y$ commute (and analogously we have that $y^2, x \circ y$ commute). Then by Remark 3.11 we have that $x^r, y^s, x \circ y$ commute by pairs for every positive integer r and s. Therefore,

$$y^q \circ (x^r \circ y^s) = x^r \circ (y^q \circ y^s) = x^r \circ y^{q+s}$$

and similarly

$$x^p \circ (x^r \circ y^s) = (x^p \circ x^r) \circ y^s = x^{p+r} \circ y^s.$$

Step 2: $R\left(x^r \circ y^s\right)$ is a polynomial in $R(x)$, $R\left(x^2\right)$, $R(y)$, $R\left(y^2\right)$, $R(x \circ y)$.
We prove the assertion for $r = 1$ by induction on s, putting $x \circ y^s = x$ for $s = 0$. Assume the assertion is true for $R(x \circ y^{s-1})$ and $R(x \circ y^s)$. Using Remark 3.10 we get the result for $s + 1$. Now we prove the assertion for a given s by induction on r. Assume that $R(x^{r-1} \circ y^s)$ and $R(x^r \circ y^s)$ are polynomials in $R(x), \ldots, R(x \circ y)$. Again using Remark 3.10, we get the assertion for $r + 1$.

Step 3: Proof of the theorem.
The hypothesis and *Step 2* imply that x^r, $x^p \circ y^q$ commute. Thus *Step 1* yields

$$(x^r \circ y^s) \circ (x^p \circ y^q) = x^r \circ ((x^p \circ y^q) \circ y^s) = x^r \circ \left(x^p \circ y^{q+s}\right) = x^{r+p} \circ y^{q+s}.$$

Therefore $F[x, y]$ is closed with respect to multiplication and is associative. $\square$

Define the quadratic representation of an algebra $\mathcal{A}$ as

$$P(x)y = 2x \circ (x \circ y) - x^2 \circ y$$

for $x, y \in \mathcal{A}$. Which can be written in terms of linear transformations as

$$P(x) = 2R^2(x) - R\left(x^2\right).$$

If $\mathcal{A}$ has unit element e then $P(e) = \mathrm{Id}$, where Id is the identity transformation.

Theorem 3.19. *Let $\mathcal{A}$ be a commutative algebra over (an infinite) field F with a unit element e. If $\mathrm{char}(F) \neq 2, 3, 5$ then $\mathcal{A}$ is a Jordan algebra if and only if $P(x^2) = P^2(x)$ holds for every $x \in \mathcal{A}$.*

Proof. By using Remark 3.10 we find that

$$3R(\left(x^2\right) R(x) = 2R(x)^3 + R(\left(x^3\right)$$

and

$$2R\left(x^3\right) R(x) + R\left(x^2\right) = 2R(x)R\left(x^2\right) R(x) + R\left(x^4\right).$$

Applying Jordan's identity to the second expression yields

$$2R\left(x^3\right) R(x) + R\left(x^2\right) = 2R(x)^2 R\left(x^2\right) + R\left(x^4\right).$$

Substituting the first into the second and rearranging gives the identity

$$R\left(x^4\right) = 4R\left(x^2\right) - 4R(x)^4 + R\left(x^2\right)^2.$$

Therefore,

$$\begin{aligned} P\left(x^2\right) &= 2R\left(x^2\right)^2 - R\left(x^4\right) \\ &= 2R\left(x^2\right)^2 - (4R\left(x^2\right) - 4R(x)^4 + R\left(x^2\right)^2) \\ &= R\left(x^2\right)^2 - 4R\left(x^2\right) R(x)^2 + 4R(x)^4 \\ &= P(x)^2. \end{aligned}$$

Hence, we have shown the forward implication of the theorem. For the reverse implication substitute $e + \lambda x$ (with $\lambda \in F$) into $P\left(x^2\right) = P(x)^2$. Now compare the λ^3 coefficients to conclude that

$$4R(x)^3 + 2R\left(x^3\right) = 3R(x)R\left(x^2\right) + 3R\left(x^2\right) R(x).$$

The assumption $P\left(x^2\right)=P(x)^2$ when applied as a linear transformation to e yields

$$x^4 = x^2 \circ x^2.$$

Replacing x with $x+\lambda y$ $(\lambda \in F)$ and comparing λ coefficients gives

$$4x^2 \circ (x \circ y) = 2x \circ (x \circ (x \circ y)) + x \circ \left(x^2 \circ y\right) + x^3 \circ y,$$

which when viewed as a linear transformation applied to y gives the identity

$$4R\left(x^2\right)R(x) = 2R(x)^3 + R(x)R\left(x^2\right) + R\left(x^3\right).$$

When combined with the previous identity we conclude that $R\left(x^2\right)R(x) = R(x)R\left(x^2\right)$ (provided that $\operatorname{char}(F) \neq 5$), which is Jordan's identity. Hence, $\mathcal{A}$ is a Jordan algebra. $\square$

For $\mathcal{A}^+$, the quadratic representation simplifies to $P(x)y = xyx$.

Theorem 3.20. *Let $\mathcal{J}$ be a Jordan algebra with a unit element e, and let $x \in \mathcal{J}$. Then the inverse x^{-1} exists if and only if $\det(P(x)) \neq 0$. In such a case we also have that $x^{-1} = P^{-1}(x)x$.*

Proof. Assume that x^{-1} exists. Then by Theorem 3.18 we have that $R\left(x^n\right)$ and $R\left(x^m\right)$ commute for integers n, m. Letting $y = x^{-1}$ and $y = x^{-2}$ in Remark 3.10 we have that

$$R(x) = P(x)R\left(x^{-1}\right) \text{ and } 2R(x)R\left(x^{-1}\right) - P(x)R\left(x^{-2}\right) = \mathrm{Id}.$$

Substituting the first into the section yields

$$P(x)P\left(x^{-1}\right) = \mathrm{Id}$$

which implies that $\det(P(x)) \neq 0$. Conversely, suppose that x^{-1} does not exist. We now show that this means that x is a zero divisor in $F[x] = \mathrm{Sp}\left(\{e, x, x^2, \dots\}\right)$. Consider the map $F[x] \to F[x]$ defined by $y \mapsto x \circ y$. Then this map is an injective endomorphism as if $x \circ y_1 = x \circ y_2$ then $x \circ (y_1 - y_2) = 0$ which means that $y_1 - y_2$ is a zero divisor as x is not a divisor, from which we deduce that $y_1 = y_2$. Hence, the map is also a bijection which implies there exists a y such that $x \circ y = e$, but this contradicts x^{-1} not existing. Therefore, there must exist a y such that $x \circ y = 0$ and hence

$$P(x)y = x \circ (x \circ y) = 0$$

which implies that $\det(P(x)) = 0$.
Consequently, we can conclude that when $\det(P(x)) \neq 0$ that

$$P^{-1}(x)x = P\left(x^{-1}\right)x = 2x^{-1} \circ \left(x^{-1} \circ x\right) - x^{-2} \circ x = x^{-1}.$$

$\square$

Proposition 3.21. *The inverse of $x \in \mathcal{A}$ exists if and only if the inverse of x in $\mathcal{A}^+$ exists.*

Proof. Let $y \in \mathcal{A}$ be such that $xy = yx = e$ then it is clear that $x \circ y = y \circ x = e$. Hence, y is the inverse of x in $\mathcal{A}^+$. Now let y be the inverse of x in $\mathcal{A}^+$. Then, $y = P^{-1}(x)x$ and hence

$$P(x)(e - xy) = x(c - xy)x = x^2 - x^2yx = x^2 - xP(x)y = x^2 - x^2 = 0.$$

Therefore, $e = xy$ and similarly $yx = e$. □

3.3 Ideals

Theorem 3.22 (The Second Isomorphism Theorem)**.** *For a Jordan algebra $\mathcal{J}$ with ideals $\mathcal{I}$ and $\mathcal{K}$ it follows that*

$$(\mathcal{I} + \mathcal{K})/\mathcal{J} \cong \mathcal{I}/(\mathcal{I} \cap \mathcal{K}).$$

Proof. Define a map $\phi : \mathcal{I} \to (\mathcal{I} + \mathcal{K})/\mathcal{J}$ by

$$\phi(i) = i + \mathcal{K}, \quad i \in \mathcal{I}.$$

Then for $i \in \mathcal{I}$ if $\phi(i) = \mathcal{K}$ we must have that $i \in \mathcal{K}$. Similarly, if $i \in \mathcal{K}$ then $\phi(i) = \mathcal{K}$. Hence, $\ker(\phi) = \mathcal{I} \cap \mathcal{K}$. Furthermore, for i_1 and $i_2 \in \mathcal{I}$ it follows that

$$\phi(i_1 + i_2) = i_1 + i_2 + \mathcal{K} = (i_1 + \mathcal{K}) + (i_2 + \mathcal{K}) = \phi(i_1) + \phi(i_2)$$

and

$$\phi(i_1 \circ i_2) = i_1 \circ i_2 + \mathcal{K} = i_1 \circ i_2 + i_1 \circ \mathcal{K} + \mathcal{K} \circ i_2 + \mathcal{K} \circ \mathcal{K} = \phi(i_1) \circ \phi(i_2).$$

Therefore, by the first isomorphism theorem, the desired relation is established. □

Proposition 3.23. *Let $\mathcal{J}$ be a Jordan algebra. Then:*

1. *If $\mathcal{J}$ is solvable, then all subalgebras of $\mathcal{J}$ are solvable.*
2. *If $\mathcal{J}$ is solvable, then all quotient algebras $\mathcal{J}/\mathcal{I}$ are solvable.*
3. *If $\mathcal{I}$ is a solvable ideal, and $\mathcal{J}/\mathcal{I}$ is solvable, then $\mathcal{J}$ is solvable.*
4. *For solvable ideals $\mathcal{I}$ and $\mathcal{K}$ it follows that $\mathcal{I} + \mathcal{K}$ is solvable.*

Proof.

1. Suppose $\mathcal{J}^{(m)} = 0$, and let $\mathcal{M} \subset \mathcal{J}$ be a subalgebra. Assume that $\mathcal{M}^{(k)} \subseteq \mathcal{J}^{(k)}$ for all $k < n$. Then, for $m \in \mathcal{M}^{(n)}$ we have that $m = x \circ y$ for some $x, y \in \mathcal{M}^{(n-1)} \subseteq \mathcal{J}^{(n-1)}$. Hence, $m \in \mathcal{J}^{(n)}$ from which it follows that $\mathcal{M}^{(m)} \subseteq \mathcal{J}^{(m)} = 0$. Therefore, $\mathcal{M}$ is also solvable.

2. Let $\mathcal{I}$ be an ideal and $\mathcal{J}^{(m)} = 0$. For $j_1, j_2 \in \mathcal{J}$ we have

$$\begin{aligned}[j_1 + \mathcal{I}, j_2 + \mathcal{I}] &= (j_1 + \mathcal{I})(j_2 + \mathcal{I}) + (j_2 + \mathcal{I})(j_1 + \mathcal{I}) \\ &= [j_1, j_2] + \mathcal{I}\end{aligned}$$

and hence, $(\mathcal{J}/\mathcal{I})^{(1)} = \mathcal{J}^{(1)}/\mathcal{I} = (\mathcal{J}^{(1)} + \mathcal{I})/\mathcal{I}$. By induction it follows that $(\mathcal{J}/\mathcal{I})^{(k)} = (\mathcal{J}^{(k)} + \mathcal{I})/\mathcal{I}$. Hence,

$$(\mathcal{J}/\mathcal{I})^{(m)} = (\mathcal{J}^{(m)} + \mathcal{I})/\mathcal{I} = 0$$

So that $\mathcal{J}/\mathcal{I}$ is solvable.

3. Assume that $\mathcal{I}^{(m)} = 0$ and $(\mathcal{J}/\mathcal{I})^{(n)} = 0$, then

$$(\mathcal{J}^{(n)} + \mathcal{I})/\mathcal{I} = \mathcal{I}$$

which implies that $\mathcal{J}^{(n)} \subseteq \mathcal{I}$. Hence,

$$0 = \left(\mathcal{J}^{(n)}\right)^{(m)} = \mathcal{J}^{(n+m)}$$

so that $\mathcal{J}$ is solvable.

4. By Theorem 3.22 it follows that

$$(\mathcal{I} + \mathcal{K})/\mathcal{I} \cong \mathcal{K}/(\mathcal{I} \cap \mathcal{K})$$

The right-hand side of which is solvable by (2) and so $\mathcal{I} + \mathcal{K}$ is solvable by part 3 of this proposition.

□

3.4 References

Introductory material for this section came from [18]. Most of the identities are from [4] along with the proof of the power associativity of a Jordan algebra. The polarization formula and other results regarding Jordan algebras came from [12]. The section on ideals was motivated by similar results for Lie algebras found in [16].

4 Solvable and Nilpotent Jordan Algebras

Definition 4.1. *An algebra (ideal) in which every element is nilpotent is called a nil algebra (ideal).*

Proposition 4.2. *A nilpotent algebra $\mathcal{A}$ is a nil algebra.*

Proof. Let $\mathcal{A}$ be nilpotent, then there exists a positive integer m such that $\mathcal{A}^m = 0$ which implies that for all $x \in \mathcal{A}$ we have that $x^m = 0$. This means that $\mathcal{A}$ is a nil algebra. □

Lemma 4.3. *Let $F[x]$ be the associative algebra generated by a non-nilpotent element x. Then, $F[x]$ contains a non-zero idempotent element, e.*

Proof. Let $f(t)$ be a non-zero polynomial of least degree such that $f(0) = 0$ and $f(x) = 0$. Let $f(t) = t^k h(t)$ where $h(0) \neq 0$ and $k \geq 1$. Then, as x is not nilpotent it must be the case that $\deg(h(t)) > 0$. Therefore, the greatest common divisor of t^k and $h(t)$ is 1, as $h(t)$ has a non-zero constant term. So by Bezout's identity, there exists polynomials $p(t)$ and $q(t)$ with $\deg(p(t)) < \deg(h(t))$ such that

$$p(t)t^k + q(t)h(t) = 1.$$

Let $g(t) = p(t)t^k$, so that $g(0) = 0$ and $e := g(x) \neq 0$. Furthermore, $\deg(g(t)) < \deg(f(t))$ as $\deg(p(t)) < \deg(h(t))$. Now, $t^k g(t) + q(t)f(t) = t^k$ so that $t^k g(t) \equiv t^k \mod f(t)$. Therefore, $g(t) \equiv 1 \mod f(t)$ and hence, $g(t)^2 \equiv g(t) \mod f(t)$. As $f(x) = 0$ we deduce that, $e^2 = g(x)^2 = g(x) = e$. □

Proposition 4.4. *Any finite-dimensional power associative algebra, $\mathcal{A}$, that is not a nil algebra, contains a non-zero idempotent e.*

Proof. Let $x \in \mathcal{A}$ not be nilpotent and consider $F[x]$. Then $F[x]$ contains an idempotent element, by Lemma 4.3, $e \neq 0$ that is also in $\mathcal{A}$. □

Let $\mathcal{J}$ be a Jordan algebra, and let $\mathcal{B}$ be a subalgebra of $\mathcal{J}$. Then the set of multiplications of elements in $\mathcal{J}$ generates an associative algebra $E(\mathcal{J})$. That is, $E(\mathcal{J})$ consists of finite sums of finite products of multiplication by elements of $\mathcal{J}$. Similarly, we define $E(\mathcal{B})$ to be the subalgebra of $E(\mathcal{J})$ that contains the finite sums of finite products of multiplication by elements of $\mathcal{B}$. On the other hand, define $\mathcal{B}^*$ to be the algebra containing all finite sums of finite products of elements in $\mathcal{B}$. There is an important distinction between these two definitions. One can see that $E(\mathcal{J}) = \mathcal{J}^*$ but note that in general $E(\mathcal{B})$ and $\mathcal{B}^*$ are not equal.

Lemma 4.5. *An ideal $\mathcal{B}$ of a Jordan algebra $\mathcal{J}$ is nilpotent if and only if $E(\mathcal{B})$ is nilpotent.*

Proof. Suppose $\mathcal{B}$ is nilpotent with index k. Then a product of $k+1$ elements $x, a_2, \ldots, a_{k+1}$ with $x \in \mathcal{J}$ and $a_2, \ldots a_{k+1} \in \mathcal{B}$ can also be seen as a product of k elements by taking $b_1 = x \cdot a_2, b_2 = a_3, \ldots, b_k = a_{k+1}$. Note that each b_i is in $\mathcal{B}$ as it is an ideal and hence this product is 0 by assumption. However, a transformation $T \in E(\mathcal{B})$ applied to $x \in \mathcal{J}$ is simply a product of $k+1$ elements. Therefore all the transformations T are 0 so that $E(\mathcal{B})$ is nilpotent with an index at most k. Now suppose that $E(\mathcal{B})$ is nilpotent with index k. Then for all multiplications S_i by elements in $\mathcal{B}$, we have that $bS_1 \dots S_k = 0$ for all $b \in \mathcal{B}$. Hence, $S_1 \dots S_k = 0$ which means that $\mathcal{B}$ is nilpotent. □

Theorem 4.6. *Let $\mathcal{B}$ be a solvable subalgebra of the Jordan algebra $\mathcal{J}$ over a field F with* $\operatorname{char}(F) \neq 2$. *Then $\mathcal{B}^*$ is nilpotent.*

Proof. For algebras $\mathcal{B}$ of size 1 we have that $\mathcal{B} = bF$. Now $b^2 = b^3 = 0$ as $\mathcal{B}$ is solvable and therefore, $R\left(b^2\right) = R\left(b^3\right) = 0$ so that by (1) we must have $R(b)^3 = 0$. This implies that $(\mathcal{B}^*)^3 = 0$ and hence $\mathcal{B}^*$ is nilpotent. Assume the result holds for solvable subalgebras of size t, and let $\mathcal{B}$ be of size $t+1$.
Claim: We have that $\mathcal{B} = \mathcal{C} + wF$ for $\mathcal{C}$ a solvable subalgebra of size t.
For a non-associative algebra $\mathcal{B}$, the product of any two elements of the quotient algebra $\mathcal{B}/\mathcal{B}^2$ is zero. Let $[w], [w_2], \dots, [w_m]$ be a basis of residue classes for $\mathcal{B}/\mathcal{B}^2$ and consider $\mathcal{C}_0 = \operatorname{Sp}(\{[w_2], \dots, [w_m]\})$, which could be 0 if $\mathcal{B}/\mathcal{B}^2$ is of dimension 1. If it is non-zero then it is closed under multiplication as the product of any two elements is 0 and it is closed under addition as elements are linearly independent of $[w]$. Therefore, $\mathcal{C}_0$ forms a subalgebra of $\mathcal{B}/\mathcal{B}^2$. Now consider the elements of the residue classes $[w_2], \dots, [w_m]$ as a subset of $\mathcal{B}$. By similar arguments, this forms a subalgebra $\mathcal{C}$ of $\mathcal{B}$. Now because $w \neq 0$ we must have that $\mathcal{B}^2 \subseteq \mathcal{C}$, so that either $\mathcal{C} = \mathcal{B}^2$ or $\mathcal{C}^2 = \mathcal{B}^2$. In either case, as $\mathcal{B}$ is solvable it follows that $\mathcal{C}$ is solvable. This finishes the proof of the claim.
Now by the inductive hypothesis $\mathcal{C}^*$ is nilpotent. For $x, y, z \in \mathcal{B}$ and we have

$$R(x \circ y)R(z) + R(y \circ z)R(x) + R(z \circ x)R(y) - R((x \circ z) \circ y) \in \mathcal{C}^*\mathcal{B}^* + \mathcal{C}^* =: \mathcal{D}.$$

By (3) we have $D(x, y, z) := R(x)R(y)R(z) + R(z)R(y)R(x) \in \mathcal{D}$. We now consider some cases:

1. If $x \in \mathcal{C}$, then $R(x)R(y)R(z) \in \mathcal{C}^*\mathcal{B}^* \subseteq \mathcal{D}$.
2. If $a = w$ and $c \in \mathcal{C}$, then

$$R(x)R(y)R(z) = D(x, y, z) - R(z)R(y)R(x) \in \mathcal{D}.$$

3. If $x = z = w$ then

$$2R(x)R(y)R(z) = D(x, y, z) \in \mathcal{D}.$$

Therefore, $R(x)R(y)R(z) \in \mathcal{C}$ for every $x, y, z \in \mathcal{B}$. Hence, $(\mathcal{B}^*)^3$ is a subalgebra of $\mathcal{C}^*\mathcal{B}^* + \mathcal{C}^*$. As $\mathcal{B}$ is an associative algebra, $(\mathcal{B}^*)^2$ is a subalgebra of $\mathcal{B}$ and $(\mathcal{B}^*)^4$ is a subalgebra of $(\mathcal{C}^*\mathcal{B}^*)^2 + \mathcal{C}^*\mathcal{B}^*$ it follows that $(\mathcal{B}^*)^4$ is a subalgebra of $\mathcal{C}^*\mathcal{B}^*$. These computations show that $(\mathcal{B}^*)^{3k+1}$ is a subalgebra of $(\mathcal{C}^*)^k\mathcal{B}^*$ for $k = 1$. Proceeding by induction it follows that this holds for all k. Therefore, choosing k to be the nilpotency index of $\mathcal{C}^*$ establishes the nilpotency of $\mathcal{B}^*$. □

Corollary 4.6.1. *A Jordan algebra $\mathcal{J}$ that is solvable is nilpotent.*

Proof. Let $\mathcal{B} = \mathcal{J}$ in Theorem 4.6 to conclude that $\mathcal{J}^*$ is nilpotent. Therefore, as $E(\mathcal{J}) = \mathcal{J}^*$ we have that $\mathcal{J}$ is nilpotent by Lemma 4.5. □

Remark 4.7. *From Proposition 4.2 it follows that a solvable Jordan algebra is a nil algebra.*

Proposition 4.8. *A Jordan algebra $\mathcal{J}$ that is nilpotent is solvable.*

Proof. Elements of $\mathcal{J}^{(1)}$ are finite sums of products of the form $a \circ x$. Suppose $\mathcal{J}^{(k)}$ are finite sums of products of the form $a_k = a \circ x_1 \circ \cdots \circ x_k = aS_1 \ldots S_k$, where S_i are right multiplications. Then the elements of $\mathcal{J}^{(k+1)}$ are finite sums of products of the form $a_k b_k$, where $a_k, b_k \in \mathcal{J}^{(k)}$. So that

$$a_k b_k = a \circ x_1 \circ \cdots \circ x_k \circ b_k = aS_1 \ldots S_k S_{k+1}.$$

Therefore, by induction $\mathcal{J}^{(m)}$ consists of the finite sum of elements of the form $aS_1 \ldots S_m$, where S_i are right multiplications. Hence, if $\mathcal{J}$ is nilpotent there exists a positive integer t such that every product of t multiplications is zero. That is, $aS_1 \ldots S_t = 0$ which implies that $\mathcal{J}^{(t)} = 0$. □

In the following $\circ$ is omitted and products are simply written in the standard way for conciseness.
Let $\mathcal{J}$ be a Jordan algebra and $\mathcal{B}$ a subalgebra. Then if $x \in \mathcal{J}$ and $x\mathcal{B} := \mathrm{Sp}(\{xb : b \in \mathcal{B}\}) \subseteq \mathcal{B}$ then it is clear that $x\mathcal{B}^* \subseteq \mathcal{B}$. Conversely, if for $x \in \mathcal{J}$ we have that $x\mathcal{B}^* \subseteq \mathcal{B}$ then it is also clear that $x\mathcal{B} \subseteq \mathcal{B}$.
For the subalgebra $\mathcal{B} = \mathcal{J}E$, where E is an idempotent linear transformation of rank equal to the dimension of $\mathcal{B}$. Define $\mathcal{T}$ to be the set of all linear transformations on $\mathcal{J}$ such that $ET = ETE$.

Lemma 4.9. *If $x \in \mathcal{J}$ then $x\mathcal{B}^* \subseteq \mathcal{B}$ if and only if $R(x) \in \mathcal{T}$.*

Proof. A quantity of $\mathcal{B}$ is of the form aE for some $a \in \mathcal{J}$. Therefore, $xb = xaE$ if and only if

$$bR(x) = aER(x) = bR(x)E = aER(x)E = aER(x)E.$$

Which happens if and only if

$$ER(x) = ER(x)E.$$

□

Lemma 4.10. *Let $R(x) \in \mathcal{T}$. Then $R\left(x^k\right) R(b)$, $R(b)R\left(x^k\right)$ and $R\left(x^k b\right)$ are in $\mathcal{T}$ for every $b \in \mathcal{B}$ and positive integer k.*

Proof. As $\mathcal{B}$ is a Jordan algebra, it satisfies the Jordan identity which implies that $R(x)R(xb) = R(x)^2 R(b)$. Therefore, because we have $R(x) \in \mathcal{T}$ it follows by Lemma 4.9 that $R(xb) \in \mathcal{T}$, which then implies that $R(b) \in \mathcal{T}$. Therefore, the case $k = 1$ is true and we can proceed by induction. Assume the result is true for $k = 1, \ldots, t$. Then,

$$\begin{aligned} R\left(x^{t+1}b\right) = R\left(\left(x^t x\right) b\right) &= R\left(x^t (xb)\right) + \left(R(b) R\left(x^t\right)\right) R(x) \\ &- \left(R\left(x^t\right) R(b)\right) R(x) - R(x)\left(R(b) R\left(x^t\right)\right) \\ &- R(x)\left(R(b) R\left(x^t\right)\right) + R(x)\left(R\left(x^t\right) R(b)\right). \end{aligned}$$

As $R\left(x^t(xb)\right) \in \mathcal{T}$ by inductive assumptions, $xb \in \mathcal{B}$ and all other terms are also in $\mathcal{B}$, it follow that $R(x^{t+1}b) \in \mathcal{T}$. Using (4)

$$\begin{aligned} R\left(x^{t+1}b\right) &= R\left(x^t\right) R(xb) + R(x) R\left(x^t b\right) + R(b) R\left(x^{t+1}\right) \\ &- \left(R\left(x^t\right) R(b)\right) R(x) - R(x)\left(R(b) R\left(x^t\right)\right), \end{aligned}$$

from which we deduce that $R(b)R(x^{t+1}) \in \mathcal{T}$. Applying (3) instead shows that $R\left(x^{t+1}\right) R(b) \in \mathcal{T}$, completing the induction. □

The following identities will be useful later on and are derived from Jordan's Identity, (2) and (5):

$$\begin{aligned} R\left(x^4\right) R\left(\left(x^2\right)\left(x^2\right)\right) &= R\left(x^2\right)^2 + 2R(x) R\left(x^3\right) - 2R(x)^2 R\left(x^2\right) \\ &= R\left(x^2\right)^2 + 2R(x)\left(3R(x^2)R(x) - 2R\left(x\right)^3\right) - 2R(x)^2 R\left(x^2\right). \end{aligned}$$

Which implies that $R\left(x^4\right) = R\left(x^2\right)^2 + 4R(x)^2 R\left(x^2\right) - 4R(x)^4$ so that

- $R\left(x^2\right)^2 = R\left(x^4\right) - 4R(x)^2 R\left(x^2\right) + 4R(x)^4$,
- $R\left(x^2\right)^2 R(b) = R\left(x^4\right) R(b) - 4R(x)^2\left(R\left(x^2\right) R(b)\right) + 4R(x)^4 R(b)$,
- $R(b) R\left(x^2\right)^2 = R(b) R\left(x^4\right) - \left(R(b) R\left(x^2\right)\right) R(x)^2 + 4R(b) R(x)^4$.

From this, we deduce that

$$R(b) R\left(x^2\right)^2 \text{ and } R\left(x^2\right)^2 R(b) \in \mathcal{T}.$$

Lemma 4.11. *Let $R(x)$ be in $\mathcal{T}$ and $b \in \mathcal{B}$, $y = x^2 b$. Then $R(y), R\left(y^2\right)$ are in $\mathcal{T}$.*

Proof. $R(y) \in \mathcal{T}$ by Lemma 4.10. By (3) we have that

$$\begin{aligned} R\left(y^2\right) &= R\left(y\left(x^2 b\right)\right) \\ &= R(y)^2 + R\left(y\left(x^2\right)\right) + R(yb) R\left(x^2\right) - R\left(x^2\right) R(y) R(b) - R(b) R(y) R\left(x^2\right). \end{aligned}$$

As $R(y) \in \mathcal{T}$ it follows that $R(y)^2 \in \mathcal{T}$. Also as $yb \in \mathcal{B}$ it follows that $R(yb)R(x^2) \in \mathcal{T}$ by Lemma 4.10. As

$$\begin{aligned} R\left(yx^2\right) &= R\left(\left(x^2 b\right) x^2\right) \\ &= 2R\left(x^2 b\right) R\left(x^2\right) + R\left(x^4\right) R(b) - R\left(x^2\right)^2 R(b) + R(b) R\left(x^2\right)^2 \end{aligned}$$

we have that

$$R\left(yx^2\right)R(b) = 2R\left(x^2b\right)\left(R(x)R(b)\right) + \left(R\left(x^4\right)R(b)\right)R(b)$$
$$- \left(R\left(x^2\right)^2 R(b)\right)R(b) - \left(R(b)R\left(x^2\right)^2\right)R(b).$$

So by Lemma 4.10 it follows that $R\left(yx^2\right)R(b) \in \mathcal{T}$.
Writing $R(y)$ as $2R(x)R(xb) + R(b)R\left(x^2\right) - 2R(x)R(b)R(x)$ it follows that

$$R\left(x^2\right)R(y)R(b) = 2R(x)\left(R\left(x^2\right)R(xb)\right)R(b) + \left(R\left(x^2\right)R(b)\right)^2$$
$$- 2R(x)\left(R\left(x^2\right)R(b)\right)R(x)R(b) \in \mathcal{T}$$

as $xb \in \mathcal{B}$. Similarly, $R(b)R(y)R\left(x^2\right) \in \mathcal{T}$ and hence $R\left(y^2\right) \in \mathcal{T}$. □

Lemma 4.12. *Let $R(y), R\left(y^2\right) \in \mathcal{T}$. Then $R\left(y^k\right) \in \mathcal{T}$ for ever positive integer k.*

Proof. This follows from Remark 3.11. □

Theorem 4.13. *Let $\mathcal{J}$ be a Jordan algebra over field F of characteristic not 2. Then if all elements of $\mathcal{J}$ are nilpotent then $\mathcal{J}$ is a solvable algebra.*

Proof. The result is clear for the algebras of size one, so suppose the result holds true for all algebras of size less than n.
If $\mathcal{J} = F[x]$, then $x^k = 0$ for some k so that $\mathcal{J}^{(k)} = F[x^{2k}]$ and is therefore solvable. So suppose instead that $\mathcal{J} \neq F[x]$ for any $x \in \mathcal{J}$ and let $\mathcal{B}$ be the maximal proper subalgebra of $\mathcal{J}$. By the inductive assumptions, it follows that $\mathcal{B}$ is solvable. Therefore, by a previous theorem, it follows that $(\mathcal{B}^*)^k = 0$ for some integer k. If $\mathcal{J}\mathcal{B} \subseteq \mathcal{B}$, then $\mathcal{B}$ is an ideal and $\mathcal{J}/\mathcal{B}$ contains only nilpotent elements (as to not contradict the maximality of $\mathcal{B}$). Therefore, by the inductive assumptions $\mathcal{J}/\mathcal{B}$ is solvable, and hence, $\mathcal{J}$ is solvable.
Suppose instead that $\mathcal{J}\mathcal{B}$ is not contained in $\mathcal{B}$. Then $\mathcal{J}(\mathcal{B}^*)^k = 0$ so that $(\mathcal{J}\mathcal{B}^*)^k \subseteq \mathcal{B}$ which implies that there exists $0 < t < k$ where

$$\mathcal{J}(\mathcal{B}^*)^t \not\subseteq \mathcal{B} \text{ and } \mathcal{J}(\mathcal{B}^*)^{t+1} \subseteq \mathcal{B}.$$

Let $x \in \mathcal{J}(\mathcal{B}^*)^t$ such that $x \notin \mathcal{B}$ and $x\mathcal{B}^* \subseteq \mathcal{B}$. If $x^2\mathcal{B} \not\subseteq \mathcal{B}$ then there is a $b \in \mathcal{B}$ such that $y = x^2b \notin \mathcal{B}$. By Lemma 4.9 and Lemma 4.11 we have that

$$y\mathcal{B}^* \subseteq \mathcal{B}, \text{ and } y^2\mathcal{B}^* \subseteq \mathcal{B}.$$

Therefore, by Lemma 4.12 there always exists an element $z \notin \mathcal{B}$ such that

$$z\mathcal{B} \subseteq \mathcal{B} \text{ and } z^2\mathcal{B} \subseteq \mathcal{B}$$

and hence,

$$z^k\mathcal{B} \subseteq \mathcal{B} \quad \forall k \in \mathbb{N}.$$

The algebra generated by z is an algebra for which $\mathcal{B}$ is an ideal. So it must be the case that $\mathcal{J}$ is equal to the algebra generated by z which is a contradiction, hence proving the theorem. □

Corollary 4.13.1. *Let $\mathcal{J}$ be a Jordan algebra over field F of characteristic not 2. If $\mathcal{J}$ is a nil algebra then $\mathcal{J}$ is nilpotent.*

This section has shown that for Jordan algebra being solvable, nilpotent and a nil algebra are equivalent properties. Note that this is not a general fact about algebras.

4.1 References

Some of the initial definitions and propositions were given by [12]. Then the results in the middle part of the section are attributed to [5]. The second half of the section was developed from [4]. Lemma 4.5 was taken from [3].

5 Decomposition of Semisimple Jordan Algebras

5.1 The Radical

Theorem 5.1. *Any finite-dimensional Jordan algebra $\mathcal{J}$ has a unique solvable ideal that contains every solvable ideal of $\mathcal{J}$.*

Proof. Let $\mathcal{R}$ be a solvable ideal of maximal dimension. Then for any other solvable ideal $\mathcal{I}$ it follows by Proposition 3.23(4) that $\mathcal{I} + \mathcal{R}$ is solvable. By the definition of $\mathcal{R}$ it must be the case that $\dim(\mathcal{I} + \mathcal{R}) \leq \dim(\mathcal{R})$ which implies that $\mathcal{I} + \mathcal{R} = \mathcal{R}$ and that $\mathcal{I} \subseteq \mathcal{R}$. □

Definition 5.2. *Let $\mathcal{A}$ be an algebra, then the unique maximal solvable ideal is called the radical, $\mathcal{R}$, of $\mathcal{A}$.*

The aim of this section is to show that if $\mathcal{R} = \{0\}$ for a Jordan algebra $\mathcal{J}$, then $\mathcal{J}$ is semisimple.

Definition 5.3. *Let $\mathcal{J}$ be a Jordan algebra. Then for $x, y \in \mathcal{J}$ let*

$$\tau(x, y) := \operatorname{tr}\left(R(x \circ y)\right).$$

This is called the trace form.

Proposition 5.4. *The trace form, τ, is a symmetric bilinear form on the vector space of a Jordan algebra $\mathcal{J}$.*

Proof. Symmetry follows from the commutativity of $\circ$, required in the definition of a Jordan algebra. Linearity follows from the linearity of R. □

Lemma 5.5. *For all $x, y, w \in \mathcal{J}$ it follows that*

$$\tau(x \circ y, w) = \tau(x, y \circ w).$$

Proof. Permuting the variables of (3) cyclically gives the identity

$$\begin{aligned} &R(y \circ z)R(x) + R(x \circ y)R(z) + R(z \circ x)R(y) \\ &\qquad = R(z)R(y)R(x) + R(x)R(y)R(z) + R(y \circ (z \circ x)). \end{aligned}$$

Hence,

$$\begin{aligned} \tau(y \circ x, z) - \tau(y, x \circ z) &= \operatorname{tr}\left(R((y \circ x) \circ z) - R(y \circ (x \circ z)\right) \\ &= \operatorname{tr}[R(z)R(y)R(x) + R(x)R(y)R(z) - R(x)R(z)R(y) \\ &\quad - R(y)R(z)R(z)] \\ &= 0. \end{aligned}$$

Therefore, $\tau(y \circ x, z) = \tau(y, x \circ z)$. □

In terms of linear transformations, the above lemma says that

$$\tau(xL(y), z) = \tau(y, zR(x)).$$

Therefore, for a sequence of multiplications S_i (right or left), it follows that

$$(xS_1 \dots S_n, y) = (x, yS'_n \dots S'_1).$$

In the future, the notation may be simplified by $T = S_1 \dots S_n$ and $T' = S'_n \dots S'_1$.

Proposition 5.6. *If $\mathcal{I}$ is an ideal of an algebra $\mathcal{J}$ on which a bilinear form, τ, is defined, then $\mathcal{I}^\perp := \{y \in \mathcal{J} : \tau(y, x) = 0,\ \forall x \in \mathcal{I}\}$ is also an ideal of $\mathcal{J}$.*

Proof. Firstly, if $y_1, y_2 \in \mathcal{I}^\perp$ we have that

$$\tau(y_1 + y_2, x) = \tau(y_1, x) + \tau(y_2, x) = 0 + 0 = 0 \quad \forall x \in \mathcal{I},$$

therefore $\mathcal{I}^\perp$ is closed under addition. Now let $x \in \mathcal{I}$ and $y \in \mathcal{I}^\perp$ and $a \in \mathcal{J}$. Then, $x \circ a$ and $a \circ x \in \mathcal{I}$ so that by application of Lemma 5.5 it follows that

$$\tau(a \circ x, y) = \tau(x \circ a, y) = 0$$

which implies that

$$\tau(x, y \circ a) = (x, a \circ y) = 0.$$

Therefore, $a \circ y$ and $y \circ a \in \mathcal{I}^\perp$. ☐

Proposition 5.7. *For a non-zero idempotent element e of a Jordan algebra $\mathcal{J}$ we have that*

$$\mathrm{tr}(R(e)) \neq 0.$$

Proof. Using (5) with $t = 2$ it follows that for an idempotent element e

$$2R(e)^3 - 3R(e)^2 + R(e) = 0.$$

Therefore, the characteristic roots of $R(e)$ are $0, 1, \frac{1}{2}$ and there is a basis of $\mathcal{J}$ such that $R(e)$ has the matrix representation

$$\begin{pmatrix} I_1 & 0 & 0 \\ 0 & \frac{1}{2}I_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If the characteristic roots were all 0 then $e = 0$, contradicting the assumptions. Hence, $\mathrm{tr}(R(e)) \neq 0$. ☐

Theorem 5.8. *For a Jordan algebra $\mathcal{J}$ we have that $\mathcal{R} = \mathcal{J}^\perp$.*

Proof. $\mathcal{J}^\perp$ is an ideal of $\mathcal{J}$ by Proposition 5.6. If $\mathcal{J}^\perp$ is not a nil ideal, then $\mathcal{J}^\perp$ would contain an idempotent element $e \neq 0$ by Proposition 4.4. However, $(e, e) = \mathrm{tr}(R(e)) \neq 0$ which would contradict the properties of the elements in $\mathcal{J}^\perp$. Hence $\mathcal{J}^\perp$ is a nil ideal meaning it is solvable which implies that $\mathcal{J}^\perp \subseteq \mathcal{R}$. Conversely, if $x \in \mathcal{R}$, then $xy \in \mathcal{R}$ for $y \in \mathcal{J}$ as $\mathcal{R}$ is an ideal. Therefore $R(x \circ y)$ is nilpotent by Remark 4.7. It follows that $(x, y) = 0$ which implies that $x \in \mathcal{J}^\perp$. Therefore, $\mathcal{R} \subseteq \mathcal{J}^\perp$. ☐

5.2 Main Theorem

Theorem 5.9. *Let $\mathcal{A}$ be a finite-dimensional algebra over F such that*

1. *there exists a non-degenerate associative trace form defined on $\mathcal{A}$, and*
2. *$\mathcal{I}^2 \neq 0$ for every ideal $\mathcal{I} \neq 0$.*

Then $\mathcal{A}$ is a semisimple algebra.

Proof. Let $\mathcal{I} \neq 0$ be a minimal ideal of $\mathcal{A}$ so that by Proposition 5.6 we have that $\mathcal{I}^{\perp}$ is an ideal of $\mathcal{A}$. Then, by Lemma 1.15 either, $\mathcal{I} \cap \mathcal{I}^{\perp} = 0$ or $\mathcal{I} \cap \mathcal{I}^{\perp} = \mathcal{I}$. Suppose $\mathcal{I} \subseteq \mathcal{I}^{\perp}$, then as $\mathcal{I}^2 \neq 0$ it must be the case that the ideal generated by $\mathcal{I}^2$ is the minimal ideal $\mathcal{I}$ which means that $\mathcal{I} = \mathcal{I}^2$. An element $s \in \mathcal{I}$ can be written as

$$s = \sum (a_i b_i) T_i \quad a_i, b_i \in \mathcal{I}$$

Therefore, for $y \in \mathcal{A}$ it follows that

$$(s, y) = \sum (a_i b_i, y T_i') = \sum (a_i, b_i(y T_i')) = 0$$

as $b_i(yT_i') \in \mathcal{I} \subseteq \mathcal{I}^{\perp}$. Therefore, $s = 0$ as $(,)$ is non-degenerate, so that $\mathcal{I} = 0$ which is a contradiction. Therefore, it must be the case that $\mathcal{I} \cap \mathcal{I}^{\perp} = 0$ and hence, $\mathcal{A} = \mathcal{I} \oplus \mathcal{I}^{\perp}$. The restriction of the trace form to $\mathcal{I}^{\perp}$ is also a non-degenerate trace form meaning (i) holds for $\mathcal{I}^{\perp}$ as well. Moreover, (ii) holds for $\mathcal{I}$ so that by induction arguments on the dimension of $\mathcal{A}$ the conclusion of the proof follows. □

Corollary 5.9.1. *Any finite-dimensional Jordan algebra $\mathcal{J}$ over a field F of characteristic 0 for which $\mathcal{R} = 0$ is semisimple.*

Proof. If $\mathcal{R} = 0$ then by Theorem 5.8 we have that the trace form is a non-degenerate bilinear form. Furthermore, if an ideal $\mathcal{I}$ of $\mathcal{J}$ is such that $\mathcal{I}^2 = 0$ then $\mathcal{I}$ is nilpotent. Hence, $\mathcal{I}$ is solvable by Proposition 4.8 which means that $\mathcal{I} \subseteq \mathcal{R} = \{0\}$ so that $\mathcal{I} = 0$. Now Theorem 5.9 can be applied to get the desired conclusion. □

5.3 References

For the part on the radical, the proof of the first theorem was inspired by a similar argument made in [16]. The second theorem is from [7]. Intermediate results made to complete the proof of this theorem were taken from multiple sources including [4],[12], and [14]. The main theorem and proof of it were given in [7].

6 Special Jordan Algebras

6.1 Algebras and Involutions

Throughout let $\mathcal{A}$ be a finite-dimensional associative algebra over a field F, where $\text{char}(F) \neq 2$. Now let $J : \mathcal{A} \to \mathcal{A}$ be an involution on $\mathcal{A}$ and $\mathcal{A}_J := \text{Sp}(\{a \in \mathcal{A} : J(a) = a\})$ (i.e. the span of the J-symmetric elements of $\mathcal{A}$). Similarly, let $\mathcal{A}_J^s := \text{Sp}(\{a \in \mathcal{A} : J(a) = -a\})$ (i.e. the span J-skew elements of $\mathcal{A}$).

Theorem 6.1. *Let $\mathcal{A}$ be a finite-dimensional and associative algebra over a field F, ($\text{char}(F) \neq 2$). Then, $\mathcal{A}_J$ is a special Jordan algebra.*

Proof. Firstly, by the linearity of $\mathcal{J}$ we have that $\mathcal{A}_J$ is a linear subspace of the vector space $\mathcal{A}$. Next let $x, y \in \mathcal{A}_J$, then

$$\begin{aligned} J(x \circ y) &= \frac{1}{2}(J(xy) + J(yx)) \\ &= \frac{1}{2}(J(y)J(x) + J(x)J(y)) \\ &= \frac{1}{2}(J(x)J(y) + J(y)J(x)) \\ &= J(x) \circ J(y) \\ &= x \circ y. \end{aligned}$$

Therefore, $\mathcal{A}_J$ is closed under the Jordan product, so that $\mathcal{A}_J \subseteq \mathcal{A}^+$ and is thus a special Jordan algebra. □

Let $\mathcal{L}$ be a linear subspace of $\mathcal{A}$, with $T : \mathcal{A} \to \mathcal{A}$ an automorphism. It is clear that $\mathcal{L}^T := \{T(a) : a \in \mathcal{L}\}$ is also a linear subspace of the same dimensions, as T is a bijection. This motivates the following proposition.

Proposition 6.2. *Consider a special Jordan algebra $\mathcal{J}$ of the associative algebra $\mathcal{A}$ and let $T : \mathcal{A} \to \mathcal{A}$ an automorphism. Then $\mathcal{J}^T$ with multiplication as in $\mathcal{J}$ is also a special Jordan algebra.*

Proof. For $x, y \in \mathcal{J}^T$ there exists elements $a, b \in \mathcal{J}$ such that $T(a) = x$ and $T(b) = y$. Therefore,

$$xy = T(a)T(b) = T(ab) = T(ba) = T(b)T(a) = yx$$

which shows that multiplication is still commutative in $\mathcal{J}^T$. Furthermore,

$$\begin{aligned} x\left(x^2 y\right) = T(a)\left(T(a)^2 T(b)\right) &= T(a)\left(T\left(a^2 b\right)\right) \\ &= T\left(a\left(a^2 b\right)\right) \\ &= T\left(a^2\left(ab\right)\right) \\ &= T(a)^2(T(a)T(b)) \\ &= x^2(xy) \end{aligned}$$

which shows that the Jordan identity is still satisfied in $\mathcal{J}^T$. It is clear that $\mathcal{J}^T \subseteq \mathcal{A}^+$, meaning it is also special. $\square$

Proposition 6.3. *Let J and K be involutions acting on an associative algebra $\mathcal{A}$ such that $K = TJT^{-1}$ for some automorphism $T : \mathcal{A} \to \mathcal{A}$. Then the corresponding special Jordan algebras $\mathcal{A}_K$ and $\mathcal{A}_J$ are isomorphic.*

Proof. Let $a \in \mathcal{A}_J$ then

$$KT(a) = TJT^{-1}T(a) = T(a)$$

so that $T(a) \in \mathcal{A}_K$ and $\mathcal{A}_J^T \subseteq \mathcal{A}_K$. Similarly, for $a \in \mathcal{A}_K$ we have

$$JT^{-1}(a) = T^{-1}KTT^{-1}(a) = T^{-1}(a).$$

Therefore, $T^{-1}(a) \in \mathcal{A}_J$ which implies that $a \in \mathcal{A}_J^T$ and hence $\mathcal{A}_K \subseteq \mathcal{A}_J^T$. $\square$

Definition 6.4. *Let $\mathcal{A}$ be an algebra over a field F, and $\mathcal{B}$ a subset of $\mathcal{A}$. Then the algebra generated by $\mathcal{B}$ is the intersection of all subalgebras of $\mathcal{A}$ that contains $\mathcal{B}$. For a subset $\mathcal{B}$, denote the algebra generated by $\mathcal{B}$ as $\overline{\mathcal{B}}$.*

The following proposition considers the transposition on matrix algebras. Which will relate closely to the algebras considered in the structure theorems stated in the next section.

Proposition 6.5. *Let J be the transposition on a matrix algebra $\mathcal{A}$ of dimension n, then the algebra generated by $\mathcal{A}_J$ equals $\mathcal{A}$.*

Proof. For $n = 1$ it is clear that $\mathcal{A}_J = \mathcal{A}$, therefore the result is trivial. For n larger than 1, consider the canonical basis

$$e_{11}, \dots, e_{ij}, \dots, e_{nn}$$

so that $J(e_{ij}) = e_{ji}$. Note that e_{ii} and $e_{ij} + e_{ji}$ are in $\mathcal{A}_J$ for all i, j. It is clear that $e_{ii}(e_{ij} + e_{ji}) = e_{ij} \in \bar{\mathcal{A}}_J$ when $i \neq j$. Hence, the canonical basis is contained within $\bar{\mathcal{A}}_J$ and so the conclusions of the proposition follow. $\square$

6.2 Simple Jordan Algebras

Theorem 6.6. *Let $\mathcal{A}$ be an associative algebra of finite dimension over a field F($\mathrm{char}(F) \neq 2$). Then, $\mathcal{A}^+$ is a special simple Jordan algebra if and only if $\mathcal{A}$ is simple.*

Proof. Let $\mathcal{I}$ be an ideal of $\mathcal{A}$. Then for $x \in \mathcal{I}$ and $a \in \mathcal{A}$ it follows that $x \circ a = \frac{1}{2}(ax + xa) \in \mathcal{I}$ which shows that $\mathcal{I}^+$ is an ideal of $\mathcal{A}^+$. Hence, if $\mathcal{A}^+$ is simple, then either $\mathcal{I}^+$ is 0 or $\mathcal{A}^+$. Therefore, $\mathcal{I} = 0$ or $\mathcal{I} = \mathcal{A}$ which implies that $\mathcal{A}$ is simple.
Now suppose that $\mathcal{A}$ is simple. If $\mathcal{A}$ is a matrix algebra then $\mathcal{A}$ has the basis

$$e_{11}, \dots, e_{ij}, \dots, e_{nn}$$

where e_{ij} is the matrix with 0s everywhere apart from the ij^{th} entry which is 1. Then if $\mathcal{I} \neq 0$ is an ideal of $\mathcal{A}^+$ and $0 \neq x \in \mathcal{I}$ it follows that

$$b = \sum_{ij} \beta_{ij} e_{ij} \quad \beta_{ij} \in F, \ i, j = 1, \ldots, n.$$

Therefore,

$$e_{ii} b + b e_{ii} = \sum_{j} \beta_{ij} e_{ij} + \sum_{k} \beta_{ki} e_{ki} \in \mathcal{I} \tag{6}$$

and,

$$e_{jj}(e_{ii} b + b e_{ii}) + (e_{ii} b + b e_{ii}) e_{jj} = \beta_{ji} e_{ji} + \beta_{ij} e_{ij} \in \mathcal{I} \quad \forall i \text{ and } j \neq i. \tag{7}$$

If $\beta_{ii} \neq 0$ for some i then by (6) we have that $2\beta_{ii} e_{ii} \in \mathcal{I}$ which implies that $e_{ii} \in \mathcal{I}$. If not then $\beta_{ji} \neq 0$ for some $i \neq j$, so that by (7) we have

$$e_{ij}(\beta_{ji} e_{ji} + \beta_{ij} e_{ij}) + (\beta_{ji} e_{ji} e_{ij}) e_{ij} = \beta_{ji} e_{ii} + \beta_{ji} e_{jj} \in \mathcal{I}$$

which is a reduction to the first case, and so again we have the conclusion that $e_{ii} \in \mathcal{I}$. For $i \neq j$ we have

- $e_{ij} = e_{ii} e_{ij} + e_{ij} e_{ii} \in \mathcal{I}$,
- $e_{ji} = e_{ji} e_{ii} + e_{ii} e_{ji} \in \mathcal{I}$, and
- $e_{jj} = e_{ij} e_{ji} + e_{ji} e_{ij} - e_{ii} \in \mathcal{I}$.

Therefore, $\mathcal{I} = \mathcal{A}^+$ so that $\mathcal{A}^+$ is simple. In the general case where $\mathcal{A}$ is any simple algebra, then over F there is a scalar extension F' of F such that $\mathcal{A}'$ is a matrix algebra. $(\mathcal{A}')^+$ is simple by the above reasoning. Noting that $(\mathcal{A}')^+ = (\mathcal{A}^+)'$ it follows that $\mathcal{A}^+$ is simple for every scalar extension F' of F. Which proves the theorem in the general case as well. □

Theorem 6.7. *Let $\mathcal{A}$ be a matrix algebra of finite dimension over a field F of characteristic not 2. If $J : \mathcal{A} \to \mathcal{A}$ is an involution on $\mathcal{A}$ then the set of $\mathcal{A}_J$ of J-symmetric elements of $\mathcal{A}$ is a special Jordan algebra, that is simple.*

Proof. Note that $\mathcal{A}_J$ has a basis

$$f_{ij} = f_{ji} = \frac{1}{2}(e_{ij} + e_{ji}) \quad i, j = 1, \ldots, n.$$

For $i, j, k, t \in 1, \ldots, n$ being distinct the multiplication in $\mathcal{A}$ is defined as follows:

- $f_{ii}^2 = \frac{1}{2}(f_{ii}^2 + f_{ii}^2) = f_{ii}^2 = f_{ii}$,
- $4f_{ij}^2 = (e_{ij} + e_{ji})^2 = e_{ij}^2 + e_{ij} e_{ji} + e_{ji} e_{ij} + e_{ji}^2 = e_{ii} + e_{jj} = f_{ii} + f_{jj}$,
- $4 f_{ij} f_{ik} = \frac{1}{2}\left((e_{ij} + e_{ji})(e_{ik} + e_{ki}) + (e_{ik} + e_{ki})(e_{ij} + e_{ji})\right) = \frac{1}{2}(e_{jk} + e_{kj}) = f_{jk}$,

- $2f_{ii}f_{ik} = \frac{1}{4}(2e_{ii}(e_{ik} + e_{ki}) + (e_{ik} + e_{ki})(2e_{ii})) = f_{ik}$,
- $f_{ii}f_{jj} = f_{ii}f_{tk} = f_{ij}f_{tk} = 0$.

Let $\mathcal{I} \neq 0$ be an ideal of $\mathcal{A}_J$. If $0 \neq b = \sum_{ij} \beta_{ij} f_{ij} \in \mathcal{I}$ then

1. $2f_{ii}b = \sum_{i \neq j} \beta_{ij} + 2\beta_{ii}f_{ii} \in \mathcal{I}$, and
2. $(2f_{ii}b)(2f_{jj}) = \beta_{ij}f_{ij} \in \mathcal{I}$, for $i \neq j$.

Therefore, if all $\beta_{ij} = 0$ then some $\beta_{ii} \neq 0$ so that by Identity 1. it follows that $f_{ii} \in \mathcal{I}$. From which we have that $2f_{ii}f_{ij} = f_{ij} \in \mathcal{I}$ for $i \neq j$. Hence there is some $f_{ij} \in \mathcal{I}$, therefore, $(2f_{ij})^2 = f_{ii} + f_{jj} \in \mathcal{I}$. Consequently, $2f_{ii}f_{ik} = f_{ik} \in \mathcal{I}$ and $(2f_{ik})^2 = f_{ii} + f_{kk}$. Hence the difference $(f_{ii} + f_{kk}) - f_{ii} = f_{kk} \in \mathcal{I}$. Therefore, $\mathcal{I}$ contains every f_{ii} and every $2f_{ii}f_{ij} = f_{ij}$. Showing that $\mathcal{I} = \mathcal{A}_J$, and that $\mathcal{A}_J$ is simple. □

Consider for a moment an arbitrary field F an involution J. Let $F_1 = F$ and let $F_i = (F_{i-1}, J)$, that is the resulting algebra when applying the Cayley-Dickson doubling process to the algebra F_{i-1} with the extension of J. Denote the F-algebra of matrices of dimension n with entries in F_i by $M_n(F_i)$. We can define an involution on this algebra to be transposing the matrix and applying the involution J component-wise. We have just shown that the set of symmetric elements with respect to this involution, which we will denote $\mathcal{H}_n(F_i)$, is a simple special Jordan algebra when $M_n(F_i)$ is an associative algebra.

Now previously we have previously seen that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are a result of the Cayley-Dickson doubling process when applied to $\mathbb{R}$. In this case $\mathcal{H}_n(\mathbb{R}), \mathcal{H}_n(\mathbb{C})$, $\mathcal{H}_n(\mathbb{H})$ and $\mathcal{H}_n(\mathbb{O})$ are the familiar Hermitian matrices (that is the matrices that are equal to their conjugate transpose). Therefore, $\mathcal{H}_n(\mathbb{R}), \mathcal{H}_n(\mathbb{C})$ and $\mathcal{H}_n(\mathbb{H})$ are simple special Jordan algebras as $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ are associative. However, $\mathbb{O}$ is not associative, so we cannot conclude anything about $\mathcal{H}_n(\mathbb{O})$ from this theorem.

6.3 References

Material for this section was developed in [5].

7 Classification of Jordan Algebras

7.1 Structure Theory

The first classification result only considers so-called formally real Jordan algebras over the reals and came from Jordan, von Neumann, and Wigner. Consider a real commutative finite-dimensional algebra that is not necessarily associative. The algebra is *formally real* if

$$x_1^2 + x_2^2 + x_3^2 + \cdots = 0$$

implies that

$$x_1 = x_2 = x_3 = \cdots = 0.$$

The algebra is a *formally real Jordan algebra* if in addition, it is also a Jordan algebra.

Theorem 7.1. *A formally real algebra that is power associative is a formally real Jordan algebra, and a formally real Jordan algebra is power associative.*

Proof. First, consider a formally real Jordan algebra and let $[x, y, z] = (xy)z - x(yz)$. Then by commutativity

$$[x, y, z] + [z, y, x] = 0, \text{ and} \tag{8}$$
$$[x, y, z] + [y, z, x] + [z, x, y] = 0. \tag{9}$$

Replace x and y in Jordan's identity by $\lambda x + \mu y + \nu z$ and w respectively. Note the terms proportional to $\lambda\mu\nu$ to conclude that

$$[zy, w, z] + [yz, w, x] + [zx, w, y] = 0. \tag{10}$$

Suppose that power associativity holds for n, m such that $n + m \leq N$. Then by (8) and (10) and $1 \leq m \leq N$ it follows that

$$\left[x^{m+1}, y, x^{N-m-1}\right] = \left[x^m, y, x^{N-m}\right] + \left[x, y, x^{N-1}\right]$$

so that

$$\left[x^p, y, x^{N-p}\right] = p\left[x, y, x^{N-1}\right] \text{ for } p = 1, \ldots, N-1.$$

In particular,

$$-\left[x, y, x^{N-1}\right] = (N-1)\left[x, y, x^{N-1}\right]$$

which implies that $[x, y, x^{N-1}] = 0$ and $[x^p, y^{N-p}] = 0$. Letting $x = y$ we have

$$x^{p+1}x^{N-p} = x^p x^{N-p+1} = xx^N = x^{N+1}.$$

Therefore power associativity holds for $n+m = N+1$, so by induction, it follows that power associativity holds in general. The proof for the other direction is more involved and requires concepts that have not been developed sufficiently in this report. For further details please refer to [2]. □

Example (Spin Factor): Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product on $\mathbb{R}^n$. Then the Jordan spin factor, $\mathfrak{S}_n$, has the underlying vector space $\mathbb{R} \oplus \mathbb{R}^n$ and acts over the real numbers. Multiplication is defined as

$$(\mathbf{v}, x) \circ (\mathbf{w}, y) := (x\mathbf{w} + y\mathbf{v}, \langle \mathbf{v}, \mathbf{w} \rangle + xy).$$

This is a formally real Jordan algebra.

Proof. Clearly, $\mathbf{v} \circ \mathbf{w} = \mathbf{w} \circ \mathbf{v}$. On the one hand,

$$\begin{aligned}((\mathbf{v}, x) \circ (\mathbf{w}, y)) \circ (\mathbf{v}, x)^2 &= (x\mathbf{w} + y\mathbf{v}, \langle \mathbf{v}, \mathbf{w} \rangle + xy) \circ \left(2x\mathbf{v}, |\mathbf{v}|^2 + x^2\right) \\ &= \Big(\left[2x\langle \mathbf{v}, \mathbf{w} \rangle + 3x^2 y + y|\mathbf{v}|^2\right] \mathbf{v} + \left[x^3 + x|\mathbf{v}|^2\right] \mathbf{w}, \\ &\quad 3x^2\langle \mathbf{v}, \mathbf{v} \rangle + 3xy|\mathbf{v}|^2 + x^3 y + |\mathbf{v}|^2 \langle \mathbf{v}, \mathbf{w} \rangle\Big)\end{aligned}$$

and on the other hand,

$$\begin{aligned}\left(\mathbf{v}, x) \circ ((\mathbf{w}, y) \circ (\mathbf{v}, x)^2\right) &= (\mathbf{v}, x) \circ \left((\mathbf{w}, y) \circ \left(2x\mathbf{v}, |\mathbf{v}|^2 + x^2\right)\right) \\ &= (\mathbf{v}, x) \circ \left(2xy\mathbf{v} + \left(|\mathbf{v}|^2 + x^2\right) \mathbf{w}, \langle \mathbf{w}, 2x\mathbf{v} \rangle + y|\mathbf{v}|^2 + x^2 y\right) \\ &= \Big(\left[2x\langle \mathbf{v}, \mathbf{w} \rangle + 3x^2 y + y|\mathbf{v}|^2\right] \mathbf{v} + \left[x^3 + x|\mathbf{v}|^2\right] \mathbf{w}, \\ &\quad 3x^2\langle \mathbf{v}, \mathbf{v} \rangle + 3xy|\mathbf{v}|^2 + x^3 y + |\mathbf{v}|^2 \langle \mathbf{v}, \mathbf{w} \rangle\Big).\end{aligned}$$

Therefore, Jordan's identity is satisfied. Note that $|\mathbf{v}|^2 = \langle \mathbf{v}, \mathbf{v} \rangle > 0$ for $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^n$. If,

$$(\mathbf{v}_1, x_1)^2 + \ldots (\mathbf{v}_n, x_n)^2 = 0$$

then

$$\left(2x_1\mathbf{v}_1, x_1^2 + |\mathbf{v}_1|^2\right) + \cdots + \left(2x_n\mathbf{v}_n, x_n^2 + |\mathbf{v}_n|^2\right) = 0.$$

By comparing the $\mathbb{R}$ component, it is clear that this can only happen if $x_i = 0$ and $\mathbf{v}_i = \mathbf{0}$ for all i. □

Consider the basis $(0, 1)$ and $(\mathbf{e}_i, 0)$ for $i = 1, \ldots n$, where $\mathbf{e}_i$ is the i^{th} standard basis vector of $\mathbb{R}^n$. Then the defining relations for multiplication on $\mathfrak{S}_n$ are

$$(\mathbf{e}_i, 0) \circ (\mathbf{e}_i, 0) = (0, 1), \quad (0, 1) \circ (0, 1) = 1, \quad (\mathbf{e}_i, 0) \circ (\mathbf{e}_j, 0) = 0 \quad \text{for } i \neq j.$$

Therefore, one can equivalently define $\mathfrak{S}_n$ as the algebra over $\mathbb{R}$ with the linear basis $1, s_1, \ldots, s_n$. In which $a \pm b$ and λa are defined in the usual way for $a, b \in \mathfrak{S}_n$ and $\lambda \in F$ but multiplication is defined by the relations

$$11 = 1, \quad 1s_i = s_i, \quad s_i s_j = \delta_{ij} 1.$$

It was under these conditions that Jordan was able to give the first classification theorem of Jordan algebras.

Theorem 7.2. *The only simple formally real Jordan algebras over the reals are:*

- $\mathbb{R}$,
- $\mathfrak{S}_n$ *for* $n \geq 2$,
- $\mathcal{H}_3(K_i)$, $i = 1, 2, 3, 4$,
- $\mathcal{H}_n(K_i)$, $i = 1, 2, 3$ *and* $n \geq 4$.

Where K_1, K_2, K_3 *and* K_4 *correspond to* $\mathbb{R}, \mathbb{C}, \mathbb{H}$ *and* $\mathbb{O}$.

Theorem 7.3. *All the Jordan algebras in the above list, with the exception of* $\mathcal{H}_3(\mathbb{O})$*, are special.*

Proof. This is clear for $\mathbb{R}$. For $\mathcal{H}_n(K_i)$ $(i = 1, 2, 3)$, it is also clear as the reals, complexes, and quaternions are associative algebras. For $\mathfrak{S}_n$ we show that it arises by defining the Jordan product on the system in which 1 is the identity, $s_i^2 = 1$, and $s_i s_j = -s_j s_i$ $(i \neq j$ and $i, j = 1, \ldots, n)$. We have that $1 \circ 1 = \frac{1}{2}(11 + 11) = 1$ and $1 \circ s_i = \frac{1}{2}(1s_i + s_i 1) = s_i$. Furthermore,

$$s_i s_j = \begin{cases} \frac{1}{2}(s_i^2 + s_i^2) = 1 = \delta_{ij} 1 & i = j \\ \frac{1}{2}(s_i s_j + s_j s_i) = \frac{1}{2}(s_i s_j - s_i s_j) = 0 = \delta_{ij} 1 & i \neq j \end{cases}.$$

Hence, the multiplication rules are satisfied. Proof that $\mathcal{H}_3(\mathbb{O})$ is not special is developed in the next subsection. □

Work was done by A. Albert to extend these results to deal with finite-dimensional Jordan algebras over arbitrary fields. A corollary of these results includes the classification of finite-dimensional Jordan algebras over algebraically closed fields.

Theorem 7.4. *Let* $\mathcal{J}$ *be a finite-dimensional simple Jordan algebra over an algebraically closed field* F*. Then one of the following holds:*

1. $\mathcal{J} = F$
2. $\mathcal{J} = F \oplus \mathcal{B}$*, the Jordan algebra of a non-degenerate bilinear form in a vector space* $\mathcal{B}$ *where* $\dim(\mathcal{B}) > 1$.
3. $\mathcal{J} = \mathcal{H}_n(F_i)$*, for* $i = 1, 2, 3$ *and* $n = 3$.
4. $\mathcal{J} = \mathcal{H}_n(F_i)$*, for* $i = 1, 2, 3, 4$ *and* $n \geq 4$.

Remark 7.5. *Note that this provides a classification of semisimple Jordan algebras, as semisimple Jordan algebras can be written as a direct sum of simple algebras.*

Extensions of these results to arbitrary fields were made by Kalisch, F.D Jacobson, and N. Jacobson. Later work in 1979 by Zelmanov had significant implications on the original motivations of Jordan algebras. Zelmanov showed that there were no new exceptional algebras in infinite dimensions. Therefore, Jordan's proposed framework could not accommodate quantum mechanics as intended. Zelmanov also gave a complete classification of Jordan algebras over division algebras.

7.2 The Exceptional Jordan Algebra

The octonions, $\mathbb{O}$, are a real algebra with units $1(= e_0), e_1, \ldots, e_7$, and multiplication rule:

$$e_i e_j = \begin{cases} e_j & e_i = 1 \\ e_i & e_j = 1 \\ -\delta_{ij} 1 + \epsilon_{ijk} e_k & \text{otherwise.} \end{cases}$$

Within the octonions, there are systems that are equivalent to the quaternions:

- $(1, e_1, e_2, e_3)$,
- $(1, e_1, e_4, e_5)$,
- $(1, e_1, e_6, e_7)$,
- $(1, e_2, e_5, e_7)$,
- $(1, e_2, e_4, -e_6)$, and
- $(1, e_3, e_5, e_6)$.

For $a = \lambda_0 + \sum_{i=1}^{7} \lambda_i e_i$ define:

- $\bar{a} = \lambda_0 - \sum_{i=1}^{7} \lambda_i e_i$,
- $T(a) = a + \bar{a} = 2\lambda_0$, and
- $N(a) = a\bar{a} = \sum_{i=0}^{7} \lambda_i^2$.

From which it follows that:

- $w^2 - T(a)w + N(a) = 0$,
- $\overline{ab} = \bar{b}\bar{a}$,
- $N(ab) = N(a)N(b)$,
- $T(\lambda a + \mu b) = \lambda T(a) + \mu T(b)$ for $\lambda, \mu \in \mathbb{R}$, and
- $a(\bar{a}b) = (a\bar{a})b$.

Theorem 7.6. *Let i be any unit of $\mathbb{O}$ and let $a, b \in \mathbb{O}$. Then*

$$i(ab)i = (ia)(bi), \quad (iai)(ib) = -i(ab), \text{ and } (ai)(ibi) = -(ab)i. \tag{11}$$

Proof. Suffices to prove the result for units. If i, a, b are in a quaternion system then the results hold by application of the associative law. If not then without loss of generality let $i = e_1, a = e_2$ and $b = e_4$. Then,

$$i(ab)i = -i^2(ab) = e_2 e_4 = -e_6, \text{ and } (ia)(ib) = e_3(-e_5) = -e_6$$

which proves the first identity. Next,

$$(iai)(ib) = a(ib) = e_2 e_5 = e_7, \text{ and } -i(ab) = e_1 e_6 = e_7$$

proves the second identity. Finally,

$$(ai)(ibi) = (-e_3)(e_4) = -e_7, \text{ and } -(ab)i = i(ab) = -e_7$$

proves the third identity and completes the proof. □

Theorem 7.7. *For $j = 0, \ldots, 7$ the transformations which replace e_j by $ie_j,\ e_ji,\ ie_ji$ for a unit $i \neq 1$ are orthogonal transformations.*

Proof. Consider

$$a = \sum_{j=0}^{7} \lambda_j e_j, \quad b = \sum_{k=0}^{7} \mu_k e_k,$$

so that

$$T(a\bar{b}) = \sum_{j,k} \lambda_j \mu_k T(e_j \bar{e}_k).$$

As

$$e_j \bar{e}_k = \begin{cases} \neq \pm 1 & j \neq k \\ 1 & j = k \end{cases}$$

it follows that

$$T(e_j \bar{e}_k) = \begin{cases} 0 & j \neq k \\ 2 & j = k. \end{cases}$$

Hence, we have that

$$T(a\bar{b}) = 2 \sum_{j=0}^{7} \lambda_j \mu_j. \tag{12}$$

Let $u_i = \pm e_i$ for $i = 0, \ldots, 7$, then

$$\frac{1}{2} T(u_j \bar{u}_k) = \begin{cases} 0 & j \neq k \\ 1 & j = k. \end{cases}$$

Therefore, the value of (12) is unchanged under such a transformation, showing the transformations are orthogonal. □

Let

$$\mathcal{H}_3(\mathbb{O}) = \left\{ A = \begin{pmatrix} \alpha & a & \bar{c} \\ \bar{a} & \beta & b \\ c & \bar{b} & \gamma \end{pmatrix} = A[(\alpha, \beta, \gamma), (a, b, c)] : \alpha, \beta, \gamma \in \mathbb{R} \text{ and } a, b, c \in \mathbb{O} \right\}.$$

For $\delta \in \mathbb{R}$ and $B = A[(\lambda, \mu, \nu), (f, g, h)]$ note that

$$\begin{aligned} A + B &= A[(\alpha + \lambda, \beta + \mu, \gamma + \nu), (a + f, b + g, c + h)], \text{ and} \\ \delta A &= A[(\delta\alpha, \delta\beta, \delta\gamma), (\delta a, \delta b, \delta c)]. \end{aligned}$$

Furthermore,

$$AB = BA = \frac{1}{2}(A \cdot B + B \cdot A) = A[(\xi, \eta, \zeta), (p, q, r)]$$

where $A \cdot B$ denotes regular matrix product and

$$\begin{array}{rl} \xi = \alpha\lambda + \frac{1}{2}T(a\bar{f}) + \frac{1}{2}T(\bar{c}h), & 2p = (\alpha + \beta)f + (\lambda + \mu)a + \bar{c}\bar{g} + \bar{h}\bar{b} \\ \eta = \beta\mu + \frac{1}{2}T(b\bar{g}) + \frac{1}{2}T(\bar{a}f), & 2q = (\beta + \gamma)g + (\mu + \nu)b + \bar{a}\bar{h} + \bar{f}\bar{c} \\ \zeta = \gamma\nu + \frac{1}{2}T(c\bar{h}) + \frac{1}{2}T(\bar{b}g), & 2p = (\gamma + \alpha)h + (\nu + \lambda)c + \bar{b}\bar{f} + \bar{g}\bar{a}. \end{array} \tag{13}$$

Therefore, for $A^2 = A[(\alpha_0, \beta_0, \gamma_0), (a_0, b_0, c_0)]$ we have that

$$\begin{array}{rl} \alpha_0 = \alpha^2 + N(a) + N(c), & a_0 = (\alpha + \beta)a + \bar{c}\bar{b} \\ \beta_0 = \beta^2 + N(b) + N(a), & b_0 = (\beta + \gamma)b + \bar{a}\bar{c} \\ \gamma_0 = \gamma^2 + N(c) + N(b), & c_0 = (\gamma + \alpha)c + \bar{b}\bar{a}. \end{array} \tag{14}$$

Theorem 7.8. *For every $A \in \mathcal{H}_3(\mathbb{O})$ and $B = A[(\lambda, \mu, \nu), (0, 0, 0)]$ it follows that*

$$(AB)A^2 = A(BA^2).$$

Proof. Consider the map

$$A = A[(\alpha, \beta, \gamma), (a, b, c)] \leftrightarrow A^* = [(\alpha, \beta, \gamma), (a', b'', c''')]$$

where $a' = iai, b'' = ib, c''' = ci$ for any unit i. The mapping is preserved under addition, real scalar multiplication, and under multiplication. Hence, it defines an automorphism on $\mathcal{H}_3(\mathbb{O})$. The first two preservations are clear, but multiplication requires justification. Let

$$\begin{array}{r} (AB)^* = A[(\xi, \eta, \zeta), (p', q'', r''')], \text{ and} \\ A^*B^* = A[(\xi^*, \eta^*, \zeta^*), (p^*, q^*, r^*)]. \end{array}$$

By Theorem 7.7 the transformations of units are orthogonal so the values of $T(\cdot)$ remain unchanged. It is clear from (13) that $\xi = \xi^*$, $\eta = \eta^*$ and $\zeta = \zeta^*$. Next,

$$(\bar{c}\bar{g})' = i(\bar{c}\bar{g})i = (\bar{c}\bar{g})^* = (\bar{c}\bar{i})(\bar{i}\bar{g}) = (i\bar{c})(\bar{g}i) = i(\bar{c}\bar{g})i$$

by Theorem 7.6, hence, $p' = p^*$. Similarly, $q'' = q^*$ and $r''' = r^*$ by applications of Theorem 7.6. □

Lemma 7.9. *Let $F(B, D, \ldots, K)$ be a polynomial in variables $B, D, \ldots, K$ in $\mathcal{H}_3(\mathbb{O})$, let F have real coefficients and be linear in B, and let $F = 0$ for any real matrix $B \in \mathcal{H}_3(\mathbb{O})$. Then $F = 0$ for all $B, \ldots, K$ of $\mathcal{H}_3(\mathbb{O})$.*

Proof. Let $B = A[(0, 0, 0), (0, 0, 1)]$, then $B^* = A[(0, 0, 0), (0, 0, i)]$ (using the automorphism in the proof of Theorem 7.8). As $F = 0$ for B, it follows that $F = 0$ for any such B^*. As F is linear in B, and elements of $\mathbb{O}$ are linear combinations of the units, it follows that $F = 0$ for $B = A[(0,0,0),(0,0,c)]$ where $c \in \mathbb{O}$. Permuting elements cyclically it follows that $F = 0$ for

$$A[(0,0,0),(0,b,0)], \text{ and } A[(0,0,0),(a,0,0)].$$

By assumption $F = 0$ for $B = A[(\alpha,\beta,\gamma),(0,0,0)]$ so using the above observations it follows that $F = 0$ for arbitrary $B \in \mathcal{H}_3(\mathbb{O})$. □

Lemma 7.10. *Let $A^* = U \cdot A \cdot U^{-1}$ for*

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad U' = U^{-1}$$

a real orthogonal matrix. Then $A^* \in \mathcal{H}_3(\mathbb{O})$ and $A \leftrightarrow A^*$ generates an automorphism.

Proof. We have that A^* is hermitian when A is hermitian so that $A^* \in \mathcal{H}_3(\mathbb{O})$. Furthermore,

$$(A+B)^* = A^* + B^*, \text{ and } \quad (\delta A)^* = \delta A^*.$$

Next note that

$$(AB)^* = \frac{1}{2}\left[(U \cdot (A \cdot B) \cdot U^{-1}) + U \cdot (B \cdot A) \cdot U^{-1}\right]$$

and $U \cdot (A \cdot B) \cdot U^{-1} = A^* \cdot B^*$ because U is real and matrix multiplication of matrices with only two non-real components is associative. Therefore,

$$(AB)^* = \frac{1}{2}(A^* \cdot B^* + B^* \cdot A^*) = A^* B^*.$$

□

Theorem 7.11. *For every A and B of $\mathcal{H}_3(\mathbb{O})$ we have $(AB)A^2 = A(BA^2)$ so that $\mathcal{H}_3(\mathbb{O})$ is a formally real Jordan algebra.*

Proof. Using the automorphism defined in Lemma 7.10 and Theorem 7.8 we can show that $F = (AB)A^2 - A(BA^2)$ satisfies the conditions of Lemma 7.9 which will then prove the theorem. To see this note that

$$B_1 = A[(0,1,-1),(0,0,0)] \leftrightarrow B_1^* = A[(0,0,0),(0,0,1)]$$

so that $F = 0$ for B_1 and it follows that $F = 0$ for B_1^*. Therefore, for $B = A[(0,0,0),(a,b,c)]$ we have $F = 0$. Using a similar method of proof as Lemma 7.9 it follows that $F = 0$ for any $B \in \mathcal{H}_3(\mathbb{O})$. □

Theorem 7.12. *There exists no associative algebra $\mathcal{A}$ for which $\mathcal{H}_3(\mathbb{O})$ is isomorphic to a subalgebra of $\mathcal{A}^+$. Thus, the algebra $\mathcal{H}_3(\mathbb{O})$ is an exceptional Jordan algebra.*

Proof. Suppose that $\mathcal{H}_3(\mathbb{O})$ is derived from an algebra $\mathcal{A}$. Let,

$$J_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$E_i = \begin{pmatrix} 0 & e_i & 0 \\ \bar{e}_i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, F_j = \begin{pmatrix} 0 & 0 & e_j \\ 0 & 0 & 0 \\ \bar{e}_j & 0 & 0 \end{pmatrix}, G_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_k \\ 0 & \bar{e}_k & 0 \end{pmatrix}$$

for $i, j, k = 0, \ldots, 7$ be a basis for $\mathcal{H}_3(\mathbb{O})$. Let $\mathcal{A}$ have the corresponding basis $K_1, K_2, K_3, P_i, Q_j, R_k$. That is these matrices form a basis on the underlying vector space with the normal notion of matrix multiplication. As the J_i are each idempotent it follows by Proposition 3.12 that each K_i is idempotent. Furthermore,

$$J_1 J_2 = \frac{1}{2}(J_1 \cdot J_2 + J_2 \cdot J_1) = 0$$

so that $K_1 K_2 = 0$. Note that

$$K_1 K_2 = \frac{1}{2}(K_1 \cdot K_2 + K_2 \cdot K_1)$$

so that $K_1 \cdot K_2 + K_2 \cdot K_2 = 0$. However,

$$\begin{aligned} K_1 \cdot (K_1 \cdot K_2 + K_2 \cdot K_1) \cdot K_1 &= K_1 \cdot (K_2 \cdot K_1 + K_1 \cdot K_2) \cdot K_1 \\ &= K_1 \cdot K_2 \cdot K_1^2 + K_1^2 \cdot K_2 \cdot K_1 \\ &= 2K_1 \cdot K_2 \cdot K_1 \end{aligned}$$

which implies that

$$0 = K_1 \cdot (K_1 \cdot K_2 + K_2 \cdot K_1) = K_1^2 \cdot K_2 + K_1 \cdot K_2 \cdot K_1 = K_1 \cdot K_2.$$

Similarly, $K_i \cdot K_j = 0$ when $i \neq j$ and $i, j = 1, 2, 3$. Choose the basis of $\mathcal{A}$ so that

$$K_1 = \begin{pmatrix} L_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad , K_2 = \begin{pmatrix} L_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} L_3 & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$L_1 = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

and I_1, I_2, I_3 are identity matrices of r, s, t rows respectively. As $J_1 + J_2 + J_3$ is the identity of $\mathcal{H}_3(\mathbb{O})$ it follows that $K = K_1 + K_2 + K_3$ is the identity of $\mathcal{A}$. Hence, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{A}$ we have

$$KA = \frac{1}{2}(K \cdot A + A \cdot K) = \frac{1}{2}(2A) = A,$$

but,

$$KA = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}c & 0 \end{pmatrix}$$

so that $b = c = d = 0$. Therefore, without loss of generality the elements of $\mathcal{A}$ can be taken to be square matrices of dimension $r + s + t$ which implies that $K_i = L_i$ for $i = 1, 2, 3$. Consider,

$$E = \sum_{i=0}^{7} \lambda_i E_i \in \mathcal{H}_3(\mathbb{O})$$

so that by (13) we have

$$2J_1 E = 2J_2 E = E.$$

If

$$P = \sum_{i=0}^{7} \lambda_i P_i \in \mathcal{A}$$

then it must also be the case that $2L_1 P = 2L_2 P = P$. Let $P = (a_{ij})$ then,

$$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

$$L_1 \cdot P + P \cdot L_1 = \begin{pmatrix} 2a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix},$$

$$L_2 \cdot P + P \cdot L_2 = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 2a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}.$$

Therefore,

$$P_i = \begin{pmatrix} 0 & p_i & 0 \\ p_{i0} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where p_i is a real matrix of r rows and s columns, with p_{i0} being a real matrix of s rows and r columns. Similarly,

$$Q_j = \begin{pmatrix} 0 & 0 & q_j \\ 0 & 0 & 0 \\ q_{j0} & 0 & 0 \end{pmatrix}, \quad R_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r_k \\ 0 & r_{k0} & 0 \end{pmatrix}.$$

Again by (13) we deduce that

$$2E_i E_j = \begin{pmatrix} e_i \bar{e}_j + e_j \bar{e}_i & 0 & 0 \\ 0 & \bar{e}_i e_j + \bar{e}_j e_i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$2P_iP_j = \begin{pmatrix} p_ip_{j0} + p_jp_{i0} & 0 & 0 \\ 0 & p_{i0}p_j + p_{j0}p_i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As $e_i\bar{e}_i = 1$ it follows that $2E_iE_j = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which implies that $p_ip_{i0} = I_1$ and $p_{i0}p_i = I_2$ for $i = 0, \ldots, 7$. Similar relations arise when investigating Q_j and R_k. Moreover, for $i \neq j$ as $e_i\bar{e}_j + e_j\bar{e}_i = \bar{e}_ie_j + \bar{e}_je_i = 0$ it follows that $p_ip_{j0} = -p_jp_{i0}$ for $i, j = 0, \ldots, 7$. Now

$$2E_iG_j = \begin{pmatrix} 0 & 0 & e_ie_j \\ 0 & 0 & 0 \\ \bar{e}_j\bar{e}_i & 0 & 0 \end{pmatrix} = \lambda_{ij}F_k \text{ where } e_ie_j = \lambda_{ij}e_k \in \mathbb{O}.$$

Hence,

$$\lambda_{ij}Q_k = 2P_iR_j = \begin{pmatrix} 0 & 0 & p_ir_j \\ 0 & 0 & 0 \\ r_{j0}p_{i0} & 0 & 0 \end{pmatrix}$$

from which it is deduced that

$$p_ir_j = \lambda_{ij}q_k, \quad r_{jo}p_{i0} = \lambda_{ij}q_{k0} \quad i, j = 0, \ldots, 7. \tag{15}$$

- With $i = 1$ it follows that $e_j = \lambda_{0j}e_k$ which implies that $\lambda_{0j} = 1$ and $k = j$.
- With $j = 0$, $e_ie_0 = \lambda_{i0}e_k$ so that $e_i = \lambda_{i0}e_k$ which implies $\lambda = 1$ and $i = k$ and therefore, $p_ir_0 = q_i$.

Combining both of these it follows that $p_0r_i = p_ir_0$. Multiply both on the left by p_{00} to obtain $r_i = (p_{00}p_i)r_0$. Use $q_i = p_ir_0$ from (15) to deduce that

$$p_ir_j = p_i(p_{00}p_j)r_0 = \lambda_{ij}q_k = \lambda_{ij}p_kr_0$$

and multiply on the right by r_{00} to obtain $p_i(p_{00}p_j) = \lambda_{ij}p_k$. Let $u_i = p_ip_{00}$, then u_i is a real valued r-rowed non-zero square matrix, with $u_0 = I_1$. The set of square matrices u_i forms a basis for the algebra generated by the u_i. However, $p_i(p_{00}p_j)p_{00} = u_iu_j = \lambda_{ij}u_k$ so this algebra has the same multiplication rules as $\mathbb{O}$. This is not possible as the square matrices are associative, whereas, $\mathbb{O}$ is not. This contradiction completes the proof. □

7.3 References

Structure theory material was developed in [2]. The spin factor example was developed from [18]. Information relating to the history of the various classification theorems along with the theorem relating to A.A. Albert's work came from [8]. The piece on Zelmanov's work came from [13]. The material on exceptional Jordan algebras was developed from [1].

8 Connection to Lie Algebras

Definition 8.1. *For an algebra $\mathcal{A}$, with multiplication denoted in the usual way, the commutator is defined as*

$$[a, b] := ab - ba$$

and the anti-commutator is defined as

$$\{a, b\} := ab + ba.$$

Therefore, a special Jordan algebra has its multiplication defined as half the anti-commutator operator on an associative algebra.

Definition 8.2. *A Lie algebra $\mathcal{A}_{\mathcal{L}}$ is a vector space V over a field F with a binary operation $[\cdot, \cdot] : V \times V \to V$ such that for all $x, y, z \in \mathcal{A}_{\mathcal{L}}$ and $a, b \in F$ the following are satisfied:*

1. $[ax+by, z] = a[x, z]+b[y, z]$ *and* $[x, ay+bz] = a[x, y]+b[x, z]$ *(Bilinearity),*
2. $[x, x] = 0$ *(Alternativity), and*
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *(Jacobi Identity).*

Proposition 8.3. *Let $\mathcal{A}$ be an associative algebra. Then the algebra defined on the same vector space as $\mathcal{A}$ with multiplication being the commutator operator, which we will denote $\mathcal{A}_{\mathcal{L}}$, is a Lie algebra.*

Proof. Bilinearity follows immediately from the fact that multiplication on $\mathcal{A}$ is bilinear. It is also clear that alternativity is satisfied. Therefore, it suffices to show that the Jacobi identity is satisfied to complete the proof. Let $x, y, z \in \mathcal{A}$ then

$$\begin{aligned}
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= x(yz - zy) - (yz - zy)x \\
&\quad + y(zx - xz) - (zx - xz)y \\
&\quad + z(xy - yx) - (xy - yx)z \\
&= x(yz) - x(zy) - y(zx) + z(yx) \\
&\quad + y(zx) - y(xz) - z(xy) + x(zy) \\
&\quad + z(xy) - z(yx) - x(yz) + y(xz) \\
&= 0.
\end{aligned}$$

□

Lemma 8.4. *The Lie algebra $\mathcal{A}_{\mathcal{L}}$ is anti-commutative.*

Proof. For $x, y \in \mathcal{A}_{\mathcal{L}}$, by $L2$ we have that $[x + y, x + y] = 0$. Therefore, by bilinearity it follows that

$$0 = [x, x] + [x, y] + [y, x] + [y, y].$$

Now applying alternativity we conclude that

$$0 = [x, y] + [y, x].$$

From which it is clear that $[x, y] = -[y, x]$, and thus $\mathcal{A}_{\mathcal{L}}$ is anti-commutative. $\square$

A triple system is similar to an algebra in that it is a vector space over a field F together with some function. For algebras, this function is a bilinear map known as multiplication and for triple systems, it is a trilinear map $(\cdot, \cdot, \cdot) : V \times V \times V \to V$.

Definition 8.5. *A Lie triple system $V_{\mathcal{L}}$ is a triple system where the trilinear map, denoted $[\cdot, \cdot, \cdot]$, satisfies:*

L1: $[u, v, w] = -[v, u, w]$,

L2: $[u, v, w] + [w, u, v] + [v, w, u] = 0$, *and*

L3: $[u, v, [w, x, y]] = [[u, v, w], x, y] + [w, [u, v, x], y] + [w, x, [u, v, y]]$

for $u, v, w, x, y \in V_{\mathcal{L}}$.

Definition 8.6. *A Jordan triple system $V_{\mathcal{J}}$ is a triple system where the trilinear map, denoted $\{\cdot, \cdot, \cdot\}$, satisfies:*

J1: $\{u, v, w\} = \{w, v, u\}$, *and*

J2: $\{u, v, \{w, x, y\}\} = \{\{u, v, w\}, x, y\} - \{w, \{v, u, x\}, y\} + \{w, x, \{u, v, y\}\}$

for $u, v, w, x, y \in V_{\mathcal{J}}$.

Over an associative algebra, we use $(xyz) = xyz + zyx$ to deduce the following ternary compositions.

1. $[[x, y], z] = xyz + zyx - yxz - zxy = (xyz) - (yxz)$,
2. $\{\{x, y\}, z\} = xyz + zyx + yxz + zxy = (xyz) + (yxz)$.

Lemma 8.7. *A Lie triple system is a subspace of an associative algebra that is closed under the ternary operation $[[x, y], z]$.*

This tells us that a Lie algebra is a Lie triple system with respect to the trilinear map $[x, y, z] \mapsto [[x, y], z]$

Lemma 8.8. *A Jordan triple system is a subspace of an associative algebra that is closed under the ternary operation $\{\{x, y\}, z\}$.*

This tells us that a special Jordan algebra is a Jordan triple system with respect to the trilinear map $\{x, y, z\} \mapsto \{\{x, y\}, z\}$

8.1 Triple Systems and Algebras

In this section, we explore the connections between Jordan triple systems, Lie triple systems, special Jordan algebras, and Lie algebras. First, we shall consider the behaviour of mappings in Lie and Jordan triple systems.

Lemma 8.9. *Let $V_{\mathcal{L}}$ be a Lie triple system. Let $L_{u,v} : V_{\mathcal{L}} \to V_{\mathcal{L}}$ be the linear map $L_{u,v}(w) = [u, v, w]$. The vector space $S = \mathrm{Sp}(\{L_{u,v} : u, v \in V_{\mathcal{L}}\})$ is a Lie algebra.*

Proof. Let $L_{u,v}, L_{r,s} \in S$. Then,

$$\begin{aligned}[L_{u,v}, L_{r,s}] &= L_{u,v}L_{r,s} - L_{r,s}L_{u,v} \\ &= [u, v, [r, s, \cdot]] - [r, s, [u, v, \cdot]].\end{aligned}$$

Now applying L3 we get that

$$\begin{aligned}[L_{u,v}, L_{r,s}] &= [[u, v, r], s, \cdot] + [r, [u, v, s], \cdot] \\ &= L_{[u,v,r],s} - L_{r,[u,v,s]} \in S.\end{aligned}$$

Therefore, S is closed under the commutator operation and hence forms a Lie algebra. □

Lemma 8.10. *Let $V_{\mathcal{J}}$ be a Jordan triple system. Let $J_{u,v} : V_{\mathcal{J}} \to V_{\mathcal{J}}$ be the linear map $J_{u,v}(w) = \{u, v, w\}$. The vector space $S = \mathrm{Sp}(\{J_{u,v} : u, v \in V_{\mathcal{J}}\})$ is a Lie algebra.*

Proof. Let $J_{u,v}, J_{r,s} \in S$. Then,

$$\begin{aligned}[J_{u,v}, J_{r,s}] &= J_{u,v}J_{r,s} - J_{r,s}J_{u,v} \\ &= \{u, v, \{r, s, \cdot\}\} - \{r, s, \{u, v, \cdot\}\}.\end{aligned}$$

Now applying J2 we get that

$$\begin{aligned}[J_{u,v}, J_{r,s}] &= \{\{u, v, r\}, s, \cdot\} - \{r, \{v, u, s\}, \cdot\} \\ &= J_{\{u,v,r\},s} - J_{r,\{v,u,s\}} \in S.\end{aligned}$$

Therefore, S is closed under the commutator operation and hence forms a Lie algebra. □

Now we shall consider the relation between Lie and Jordan triple systems.

Lemma 8.11. *Any Jordan triple system is a Lie triple system with respect to the triple product*

$$[x, y, z] \mapsto \{x, y, z\} - \{y, x, z\}.$$

Proof. It is sufficient to check that the Lie triple system axioms are satisfied with respect to this map. Firstly,

$$\begin{aligned}[v,u,w] &= \{v,u,w\} - \{u,v,w\} \\ &= -\left(\{u,v,w\} - \{v,u,w\}\right) \\ &= -[u,v,w]\end{aligned}$$

and therefore L1 holds. Next, note that

$$\begin{aligned}[u,v,w]+[w,u,v]+[v,w,u] &= \{u,v,w\} - \{v,u,w\} + \{w,u,v\} \\ &\quad - \{u,w,v\} + \{v,w,u\} - \{w,v,u\} \\ &= (\{u,v,w\} - \{w,v,u\}) + (\{w,u,v\} - \{v,u,w\}) \\ &\quad + (\{v,w,u\} - \{u,w,v\})\end{aligned}$$

so that by L1 we have that

$$[u,v,w]+[w,u,v]+[v,w,u] = 0+0+0 = 0$$

which shows that L2 holds. Finally,

$$[u,v,[w,x,y]] = [u,v,(\{w,x,y\} - \{x,w,y\})]$$

so by linearity we have

$$[u,v,[w,x,y]] = [u,v,\{w,x,y\}] - [u,v,\{x,w,y\}].$$

Then using our definition of $[\cdot,\cdot,\cdot]$ it follows that

$$\begin{aligned}[u,v,[w,x,y]] &= \{u,v,\{w,x,y\}\} - \{v,u,\{w,x,y\}\} \\ &\quad - \{u,v,\{x,w,y\}\} + \{v,u,\{x,w,y\}\}.\end{aligned}$$

Now by applying J2 of Jordan triple systems and appropriate grouping of terms we deduce that

$$\begin{aligned}[u,v,[w,x,y]] =&(\{\{u,v,w\},x,y\} - \{x,\{u,v,w\},y\} - \{\{v,u,w\},x,y\} + \{x,\{v,u,w\},y\}) \\ &+ (\{w,\{u,v,x\},y\} - \{\{u,v,x\},w,y\} - \{w,\{v,u,x\},y\} + \{\{v,u,x\},w,y\}) \\ &+ (\{w,x,\{u,v,y\}\} - \{x,w,\{u,v,y\}\} - \{w,x,\{v,u,y\}\} + \{x,w,\{v,u,y\}\}).\end{aligned}$$

Again by our definition of $[\cdot,\cdot,\cdot]$ it follows that

$$\begin{aligned}[u,v,[w,x,y]] =&([\{u,v,w\},x,y] - [\{v,u,w\},x,y]) \\ &+ ([w,\{u,v,x\},y] - [w,\{v,u,x\},y]) \\ &+ ([w,x,\{u,v,y\}] - [w,x,\{v,u,y\}]) \\ =&[[u,v,w],x,y] + [w,[u,v,x],y] + [w,x,[u,v,y]].\end{aligned}$$

Which establishes L3 and completes the proof. □

Finally, we construct an alternative way of transitioning from a Jordan algebra to a Jordan triple system as the proof of Lemma 8.8 has not been provided.

Definition 8.12. *Let $\mathcal{J}$ be a Jordan algebra over a field F. Then for $x, y, z \in \mathcal{J}$ define the Jordan triple product as*

$$(x, y, z) = (x \circ y) \circ z + (y \circ z) \circ x - (z \circ x) \circ y.$$

Lemma 8.13. *Any special Jordan algebra is a Jordan triple system.*

Proof. Claim: any special Jordan algebra $\mathcal{J}$ with the trilinear map $\{x, y, z\} \mapsto ((x, y, z), (y, z, x), (z, x, y))$ is a Jordan triple system where $x, y, z \in \mathcal{J}$.
For ease of notation let $X = (x, y, z), Y = (y, z, x), Z = (z, x, y)$. From the construction of our trilinear map, we have:

$$\begin{aligned}\{x, y, z\} &= ((x, y, z) \circ (y, z, x)) \circ (z, x, y) \\ &\quad + ((y, z, x) \circ (z, x, y)) \circ (x, y, z) \\ &\quad - ((z, x, y) \circ (x, y, z)) \circ (y, z, x) \\ &= (X \circ Y) \circ Z + (Y \circ Z) \circ X - (Z \circ X) \circ Y.\end{aligned}$$

Noting that each term is a cyclic permutation of the other terms we shall consider $(X \circ Y) \circ Z$. Evaluating $Z = (z, x, y)$ we have

$$(X \circ Y) \circ Z = (X \circ Y) \circ ((z \circ x) \circ y + (x \circ y) \circ z - (x \circ y) \circ z)$$

Evaluating $Y = (y, z, x)$ we have

$$\begin{aligned}(X \circ Y) \circ Z = X &\circ ((y \circ z) \circ x + (z \circ x) \circ y - (x \circ y) \circ z) \\ &\circ ((z \circ x) \circ y + (x \circ y) \circ z - (y \circ z) \circ x).\end{aligned}$$

Letting $(a)^2 := (a) \circ (a)$ and using the linearity of $\circ$ we have

$$\begin{aligned}(X \circ Y) \circ Z =& X \circ (((z \circ x) \circ y)^2 - ((y \circ z) \circ x)^2 \\ &- ((x \circ y) \circ z)^2 + 2((y \circ z) \circ x) \circ ((x \circ y) \circ z)).\end{aligned}$$

Evaluating $X = (x, y, z)$ we have

$$\begin{aligned}(X \circ Y) \circ Z =& ((x \circ y) \circ z + (y \circ z) \circ x - (z \circ x) \circ y) \\ &\circ (((z \circ x) \circ y)^2 - ((y \circ z) \circ x)^2 - ((x \circ y) \circ z)^2 \\ &+ 2((y \circ z) \circ x) \circ ((x \circ y) \circ z)).\end{aligned}$$

Again by linearity, we have

$$\begin{aligned}(X \circ Y) \circ Z =& - ((x \circ y) \circ z)^3 - ((y \circ z) \circ x)^3 - ((z \circ x) \circ y)^3 \\ &+ ((z \circ x) \circ y)^2 \circ ((x \circ y) \circ z + (y \circ z) \circ x) \\ &+ ((y \circ z) \circ x)^2 \circ ((x \circ y) \circ z + (z \circ x) \circ y) \\ &+ ((x \circ y) \circ z)^2 \circ ((y \circ z) \circ x + (z \circ x) \circ y) \\ &- 2((z \circ x) \circ y) \circ ((y \circ z) \circ x) \circ ((x \circ y) \circ z).\end{aligned}$$

As this expression is closed under cyclic permutations of x, y, z we deduce that

$$\{x, y, z\} = (X \circ Y) \circ Z + (Y \circ Z) \circ X - (Z \circ X) \circ Y = (X \circ Y) \circ Z.$$

Now it suffices to check the Jordan triple system axioms hold for the trilinear map $\{x, y, z\}$ for special Jordan algebras. J1 holds due to commutativity of $x \circ y$ and closure under cyclic permutations

$$\{x, y, z\} = (X \circ Y) \circ Z = (Y \circ Z) \circ X = (Z \circ Y) \circ X = (Z \circ Y) \circ X = \{z, y, x\}.$$

Similarly, J2 holds due to the commutativity of $x \circ y$ and closure under cyclic permutations. Let $X' = (x, y, \{z, a, b\}), Y' = (y, \{z, a, b\}, x), Z' = (z, a, b), A' = (a, b, z)$ and $B' = (b, z, a)$ then,

$$\{x, y, \{z, a, b\}\} = (X' \circ Y') \circ ((Z' \circ A) \circ B).$$

So by commutativity of $x \circ y$ and closure under cyclic permutations we can rearrange the expansion of X', Y' along with Z', A', B' to show that

$$\{x, y, \{z, a, b\}\} = \{\{x, y, z\}, a, b\} - \{z, \{y, x, a\}, b\} + \{z, a, \{x, y, b\}\}$$

completing the proof.

□

From the above lemmas, there is a natural way of moving from a special Jordan algebra to a Lie algebra. Lemma 8.13 and Lemma 8.8 give us functors from special Jordan algebras to a Jordan triple system. Lemma 8.11 gives us a functor from Jordan triple systems to Lie triple systems. To construct a functor from Lie triple systems to Lie algebras further theory on algebras must be developed. However, it is worth noting that for finite-dimensional Lie triple systems, we can construct a general functor to a Lie algebra. Thus we have constructed a natural way of mapping from special Jordan algebras to Lie algebras.

8.2 References

Definitions regarding Lie algebras were provided by [10], the definition for Jordan triple systems was taken from [9] and then [6] was used to verify some of the definitions and proofs.

References

[1] A. Adrian Albert. "On a Certain Algebra of Quantum Mechanics". In: *Annals of Mathematics* 35.1 (1934), pp. 65–73. ISSN: 0003486X. URL: http://www.jstor.org/stable/1968118 (visited on 06/01/2023).

[2] P. Jordan, J. v. Neumann, and E. Wigner. "On an Algebraic Generalization of the Quantum Mechanical Formalism". In: *Annals of Mathematics* 35.1 (1934), pp. 29–64. ISSN: 0003486X. URL: http://www.jstor.org/stable/1968117 (visited on 06/01/2023).

[3] A. A. Albert. "On Jordan Algebras of Linear Transformations". In: *Transactions of the American Mathematical Society* 59.3 (1946), pp. 524–555. ISSN: 00029947. URL: http://www.jstor.org/stable/1990270 (visited on 06/10/2023).

[4] A. A. Albert. "A Structure Theory for Jordan Algebras". In: *Annals of Mathematics* 48.3 (1947), pp. 546–567. ISSN: 0003486X. URL: http://www.jstor.org/stable/1969128 (visited on 05/28/2023).

[5] G. K. Kalisch. "On Special Jordan Algebras". In: *Transactions of the American Mathematical Society* 61.3 (1947), pp. 482–494. ISSN: 00029947. URL: http://www.jstor.org/stable/1990386 (visited on 05/30/2023).

[6] Nathan Jacobson. "Lie and Jordan Triple Systems". In: *American Journal of Mathematics* 71.1 (1949), pp. 149–170. ISSN: 00029327, 10806377.

[7] R. D. Schafer. *An Introduction to Nonassociative Algebras*. Project Gutenberg, 1961. URL: https://www.gutenberg.org/files/25156/25156-pdf.pdf.

[8] N. Jacobson. *Structure and Representations of Jordan Algebras*. Colloquium Publications. American Mathematical Society, 1968. ISBN: 9780821846407.

[9] Ottmar Loos. "Jordan triple systems, R-spaces, and bounded symmetric domains". In: *Bulletin of the American Mathematical Society* 77.4 (1971), pp. 558–561.

[10] J.E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Graduate texts in mathematics. Springer, 1972. ISBN: 9783540900528.

[11] W.K.Richolson. "A Short Proof Of The Wedderburn-Artin Theorem". In: *New Zealand Journal Of Mathematics* (1993). URL: https://www.thebookshelf.auckland.ac.nz/docs/NZJMaths/nzjmaths022/nzjmaths022-01-010.pdf.

[12] M. Koecher, A. Krieg, and S. Walcher. *The Minnesota Notes on Jordan Algebras and Their Applications*. Lecture Notes in Mathematics no. 1710. Springer, 1999. ISBN: 9783540663607. URL: https://link.springer.com/book/10.1007/BFb0096285.

[13] Kevin McCrimmon. *A Taste of Jordan Algebras*. Ed. by S. Axler, F.W. Gehring, and K.A. Ribet. Springer-Verlag, 2004.

[14] Murray R Bremner, Lúcia I Murakami, and Ivan P Shestakov. "Nonassociative algebras". In: *Handbook of Linear Algebra* (2007), pp. 69–1.

[15] ANDREI YAFAEV. *SEMISIMPLE MODULES AND ALGEBRAS*. [Online; accessed 04-2011]. 2011. URL: https://www.ucl.ac.uk/~ucahaya/SemisimpleModules.pdf.

[16] Aleksander Horawa. *M4p46: Lie Algebras*. 2016.

[17] Dr. David Helm. *M3P8 LECTURE NOTES 11: SEMISIMPLE ALGEBRAS AND ARTIN-WEDDERBURN*. [Online; accessed June-2021]. 2019. URL: https://www.ma.imperial.ac.uk/~dhelm/M3P8/notes11a.pdf.

[18] nLab authors. *Jordan algebra*. May 2023. URL: https://ncatlab.org/nlab/show/Jordan+algebra.