# Point Process for Equipment Failure Simulation

Thomas Walker
Imperial College London - Department of Mathematics

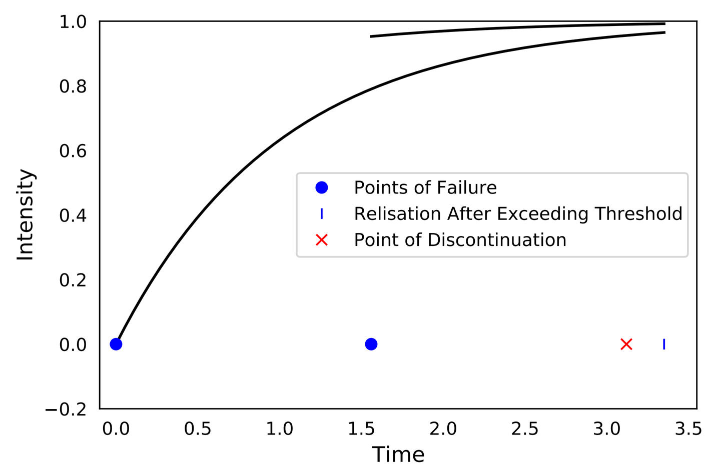
Imperial College London

## 1.1: Aim

When operators use a piece of equipment it is useful to know the potential lifespan of that piece of equipment, as well as how often that piece of equipment may fail and require repairs.
In this project I develop stochastic processes, to try and simulate the reliability of a piece of equipment.
I will leverage some work done to model earthquake shocks [1] and rainfall intensity [2], which utilise double stochastic Poisson processes.

## 2: In-homogeneous Failure Repair Model

In our first model we consider a IHPP, where at a point of failure the intensity function is shifted forward in time based on a random repair coefficient [3].

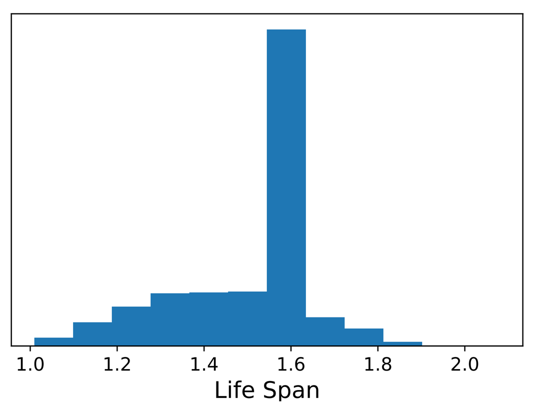



We find that its expected value should be

$$\mathbb{E}(\text{LS}) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n-1} p_k \int_0^{\infty} t\mu_k(t)\exp\left[-\int_0^t \mu_k(t)dt\right]dt + \mu_{n+1}^{-1}(\beta)\right]\left(\prod_{i=1}^{n} p_i\right)(1-p_{n+1}).$$

Where $\mu_k(t)$ is a conditional intensity, dependent on the history up to the $k^{th}$ event.

$$p_k = P(\text{k failures}) = \int_0^{\mu_k^{-1}(\beta)} \mu_k(t)\exp\left(-\int_0^t \mu_k(t)\right)dt$$




## 1.2: Definitions

- **Intensity Function**: This is a function of time and can be thought of as the rate at which the events of our stochastic process occur. Variations of which will denoted by $\mu(\cdot)$
- **In-homogeneous Poisson Process (IHPP)**: A form of stochastic process driven by a non-constant intensity function.
- **Immigration events**: These are the events caused by the underlying Poisson process. The set of these events will be denoted $\{T_i^{(0)}\}$
- **Offspring events**: These are the events caused by the Poisson process initiated by previous events in the processes history. The set of these events will be denoted $\{T_i^{(n)}\}$

## 3.1: Compounding Poisson Processes

The compounded stochastic process will be defied as $\bigcup_{k=0}^{\infty}\{T_i^{(k)}\}$. If we suppose that the times between successive immigration events follows a pdf $g(\cdot)$ it can be shown that $\mu(\cdot)$ and $g(\cdot)$ are related in the following way:

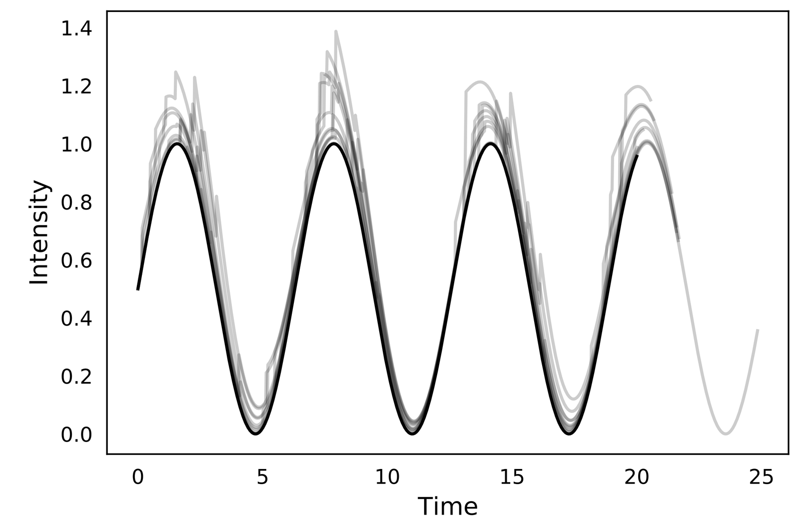
$$\mu(w) = \frac{g(w)}{1-\int_0^w g(s)ds}, \quad g(w) = \mu(w)\exp\left(-\int_0^w \mu(s)ds\right).$$

Our cumulative intensity function, CIF, can be written as:

$$\lambda(t \mid \mathcal{H}_t) = \mu(t) + \Phi(t \mid \mathcal{H}_t), \mathcal{H}_t = \{t_1, \dots, t_n \mid t_i < t\ \forall i \leq n\} \text{ denotes the history of the process}$$

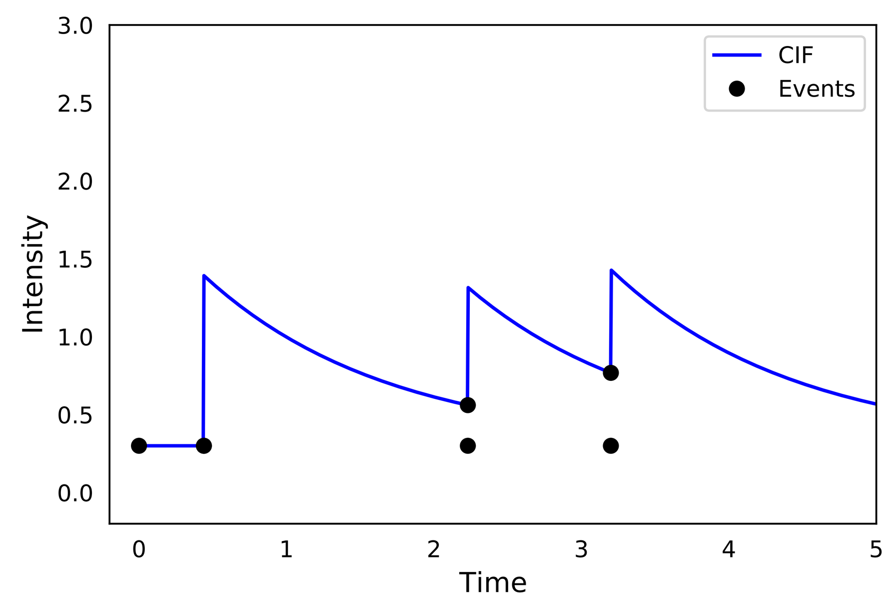
Where $\Phi(t \mid \mathcal{H}_t) = \sum_{i \leq k} \eta h(t - t_i)$, which is the cumulative intensities, caused by the off springs.
$\eta$ is a random variable for the initial offspring intensity, and $h(\cdot)$ is the offspring intensity density.
We can find the expected value of the CIF over some time interval, say a period of time in which we wish to operate the piece of equipment.

$$\mathbb{E}[\lambda(t \mid \mathcal{H}_t)] = \mathbb{E}[\mu(t)] + \mathbb{E}(\eta)\int_0^t h(u)d\mu(t-u).$$



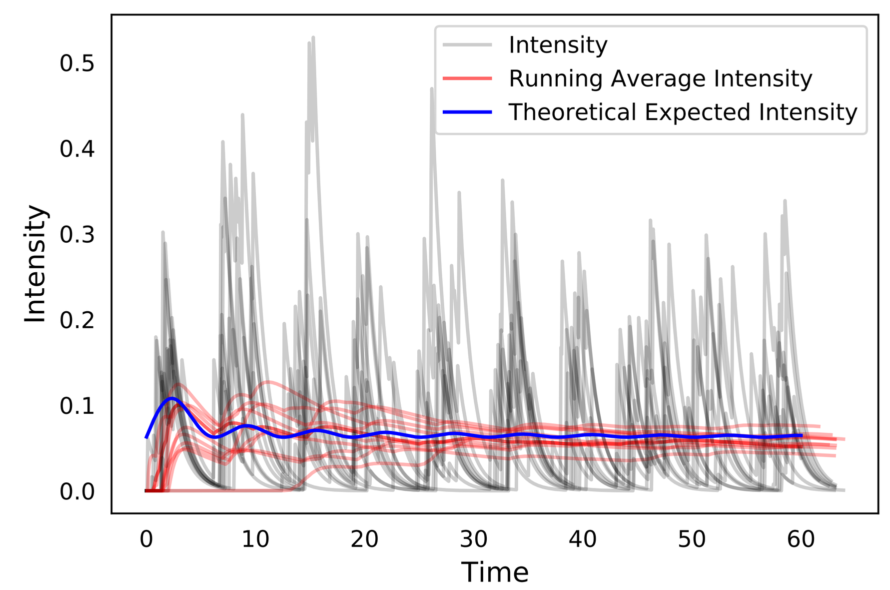

Illustrated above is the CIF for a compounded stochastic process with an underlying IHPP, driven by a sinusoidal intensity. At realisations of this process spikes of intensity decay exponentially.

## 3.2: Simulations




Here we simulate a generalisation known as the Hawke's Process. Each event triggers an IHPP, which in turn generates offspring events. Given the functional form of $\mu(\cdot)$, $h(\cdot)$ and $\eta$ we can perform maximum likelihood on this data using EM algorithms

$$\log L(\theta \mid t_{1:n}) = \sum_{i=1}^{n}\log\left[\lambda_{t_i}(t \mid \mathcal{H}_t)\right] - \int_0^r \lambda(s \mid \mathcal{H}_s)ds.$$




In this case we refine the ideas from [2], $\mu(\cdot)$ is sinusoidal, $\eta$ follows a uniform distribution and $h(\cdot) = \exp(-\beta t)$. The CIF here represents the current reliability of the piece of equipment, for a given time period we can find the expected value for this quantitative measure for unreliability

$$\mathbb{E}(\eta)\left(\int_0^t \mu(t)dt\right)\left(\frac{1-e^{-\beta t}}{\beta}\right).$$

## 4: Simulating Point Processes

To simulate these IHPP we utilise the mapping theorem [4],the Probability Integral Transform and the superposition of point processes[5]

**Mapping Theorem:** For a transformation $f\colon \mathbb{R} \to \mathbb{R}$, let $\Pi \subset \mathbb{R}$ be a Poisson process with intensity $\mu$. If $\mu^*(A) = \mu^{-1}(A)$ for $A \subset \mathbb{R}$, then the Poisson process on the $f$ transformed set $\Pi^* \subset \mathbb{R}$ is Poisson process with intensity $\mu^*$.
Using this it can be shown that for a Poisson process with intensity $\mu(t)$ that the inter arrival times have distribution

$$F_{T_i}(t) = 1 - \exp\left(-\int_{t_{i-1}}^{t} \mu(x)dx\right).$$

Which we can simulate from the Probability Integral Transform.
**Superpositon of Point Process**: For Posisson process $\Pi_1, \Pi_2 \subset \mathcal{R}$ with intensities $\mu_1(t)$ and $\mu_2(t)$ respectively, then

$$\Pi_1 \cup \Pi_2$$

is a Poisson process with intensity $\mu_1(t) + \mu_2(t)$.

## 5: References

[1] Spencer W, Vladimir F, and Didier S. The hawkes process with renewal immigration and its estimation with an em algorithm. *Computational Statistics and Data Analysis*, 94:120–135, 2016.
[2] Ramesh N, Garthwaite A, and Onof C. A doubly stochastic rainfall model with exponentially decaying pulses. *Stochastic Environmental Research and Risk Assessment*, 32:1645–1664, 2017.
[3] R. Guo and C. E. Love. Simulating nonhomogeneous poisson processes with proportional intensities. *Naval Research Logistics*, 41:516–520, 1994.
[4] Anirban DasGupta. *Probability for Statistics and Machine Learning*. Springer, New York, 1 edition, 2011.
[5] Zak Varty. Simulating poisson processes, June 2022.